

Bosons, fermions and anyons in the plane, and supersymmetry

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Abstract

Universal vector wave equations allowing for a unified description of anyons, and also of usual bosons and fermions in the plane are proposed. The existence of two essentially different types of anyons, based on unitary and also on non-unitary infinite-dimensional half-bounded representations of the (2+1)D Lorentz algebra is revealed. Those associated with non-unitary representations interpolate between bosons and fermions. The extended formulation of the theory includes the previously known Jackiw-Nair (JN) and Majorana-Dirac (MD) descriptions of anyons as particular cases, and allows us to compose bosons and fermions from entangled anyons. The theory admits a simple supersymmetric generalization, in which the JN and MD systems are unified in $N = 1$ and $N = 2$ supermultiplets. Two different non-relativistic limits of the theory are investigated. The usual one generalizes Lévy-Leblond's spin 1/2 theory to arbitrary spin, as well as to anyons. The second, "Jackiw-Nair" limit (that corresponds to Inönü-Wigner contraction with both anyon spin and light velocity going to infinity), is generalized to boson/fermion fields and interpolating anyons. The resulting exotic Galilei symmetry is studied in both the non-supersymmetric and supersymmetric cases.

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1 Introduction

According to Wigner [1], elementary particles in the plane correspond to irreducible representations of the planar Poincaré group [2]. These are labeled by two Casimir invariants, namely¹

$$(P_\mu P^\mu + m^2)\psi = 0, \quad (1.1)$$

$$(P_\mu \mathcal{J}^\mu - sm)\psi = 0, \quad (1.2)$$

where m is the mass, s the spin, and \mathcal{J}_μ is the “spin part” of the total angular momentum operator

$$\mathcal{M}_\mu = -\epsilon_{\mu\nu\lambda} x^\nu P^\lambda + \mathcal{J}_\mu. \quad (1.3)$$

\mathcal{M}_μ , together with P_μ , generate the (2+1)D Poincaré group².

For a spinless particle, $s = 0$, the second equation drops out and the Klein-Gordon equation, (1.1), is sufficient in itself.

For a Dirac particle, or for a topologically massive vector gauge field of Deser, Jackiw and Templeton (DJT) [3], i.e. for Poincaré spins $s = \pm 1/2$ or $s = \pm 1$, the Pauli-Lubanski condition, (1.2), implies (1.1) and provides, therefore, a satisfactory description. In the plane the spin can take arbitrary real values, however, and in the early nineties two, slightly different descriptions have been proposed for fractional-spin particles (called anyons [4, 5, 6, 7, 8, 9, 10, 11]). Both of them combine the half-bounded, infinite-dimensional, unitary, spin- α representations, D_α^\pm , of the Lorentz group [12] with a finite dimensional, non-unitary, “internal” representation.

Choose, for example, D_α^+ , which carries an infinite-dimensional, bounded-from-below representation of the $\mathfrak{so}(2, 1)$ algebra of the planar Lorentz group, with generators J_μ . Diagonalizing J_0 , $J_0|\alpha, n\rangle = (\alpha + n)|\alpha, n\rangle$, with $\alpha > 0$ and $n = 0, \dots, \infty$.

Jackiw and Nair (JN) [13] combine this with the spin-1 representation of topologically massive gauge theory [3]. In fact, they consider the dual to the topologically massive gauge field strength *vector* $F^\mu(x)$, which transforms under the spin-1 matrices $(j^\mu)_{\nu\lambda} = -i\epsilon_\nu{}^\mu{}_\lambda$. A “Jackiw-Nair” wave function is, hence,

$$F^\mu = \sum_{n=0}^{\infty} F_n^\mu(x) |\alpha, n\rangle. \quad (1.4)$$

Their wave equation is the Pauli-Lubanski condition (1.2), with \mathcal{J}^μ denoting the total spin [13],

$$P_\lambda(\mathcal{J}^\lambda)_{n\mu}{}_{n'\nu} F_{n'}^\nu - sm F_{n\mu} = 0, \quad \mathcal{J}^\mu = J^\mu + j^\mu, \quad s = \alpha - 1. \quad (1.5)$$

This description is redundant, however, necessitating subsidiary conditions, which reduce the number of components from 3 to 1, and generate the mass shell condition [13].

An alternative description, proposed by one of us [14] and referred to below as the “Majorana-Dirac” (MD) approach, also uses the infinite dimensional discrete type representations D_α^\pm , but the finite dimensional part is, rather, a *spinor*,

$$\psi = \sum_{n=0}^{\infty} \psi_n(x) |\alpha, n\rangle. \quad (1.6)$$

¹We took $c = 1$, so that our metric is $(-1, +1, +1)$.

²The irreducible representations of the universal covering group of $SO(2, 1)$ are, for finite-dimensional non-unitary representations, fixed uniquely by the values of the $\mathfrak{so}(2, 1)$ Casimir operator $\mathcal{J}_\mu \mathcal{J}^\mu$. Infinite-dimensional (unitary and non-unitary) representations require, in addition, to specify also the concrete series we are considering. On the other hand, the representations of the Poincaré group are labeled, in the massive case, by the Poincaré spin that corresponds to the value of the Casimir operator $P_\mu \mathcal{M}^\mu / m = P_\mu \mathcal{J}^\mu / m$.

where the $\psi_n(x)$ are two-component spinors. The posited field equations are the planar Majorana equation [15] supplemented by the planar Dirac equation,

$$(P_\mu J^\mu - \alpha m)\psi = 0, \quad (1.7)$$

$$(P_\mu \gamma^\mu - m)\psi = 0, \quad (1.8)$$

where the γ^μ are the planar Dirac (in fact Pauli) matrices. The total spin is still $\mathcal{J}^\mu = J^\mu + j^\mu$ as in (1.5), except for $j^\mu = -\gamma^\mu/2$, which carry a spin-1/2 representation, so that $s = \alpha - 1/2$. Note that in (1.7) the total spin, \mathcal{J}^μ , has been replaced by the D_α^+ generator J^μ .

The system of two equations eliminates the unphysical degrees of freedom, so that the system (1.7)-(1.8) does not necessitate any further subsidiary condition.

Both the JN and MD systems are shown to imply the Klein-Gordon equation, (1.1), and carry, therefore, an irreducible representation of the planar Poincaré group.

- Our first aim is to unify the two descriptions by replacing both of them by a suitable vector equation.

To this end we use the vector set of equations proposed earlier in [16]. By analyzing the lowest/highest weight representations of $\mathfrak{so}(2,1)$, we show that the vector set is appropriate not only to describe anyons and usual boson and fermion fields, but also provides us with an *anyon interpolation between bosons and fermions* of arbitrary spin values shifted by one half. Considering an extended formulation of the vector set of equations, we reveal that the Jackiw-Nair [13] and Majorana-Dirac [14] descriptions of anyons are incorporated in it as particular cases. By a kind of fine tuning, which results in a destructive interference, the extended formulation allows us to compose boson and fermion fields from *entangled* anyons. Both this possibility and the interpolation mechanism mentioned above rely on infinite dimensional non-unitary half-bounded representations of $\mathfrak{so}(2,1)$.

- Our second objective is to construct a supersymmetric extension of the theory.

We show that our basic vector set of equations admits a simple and natural supersymmetric generalization. The construction is based on $\mathfrak{osp}(1|2)$ representations realized in terms of a special deformation of the Heisenberg algebra [17, 18, 19, 20]. As interesting examples, we consider an $N = 1$ supermultiplet that includes the Dirac spinor and topologically massive DJT vector gauge fields, and unify the JN and MD anyon fields into $N = 1$ and $N = 2$ supermultiplets.

- Finally, two different non-relativistic limits are investigated both in the supersymmetric and the non-supersymmetric cases.

The first, usual, limit yields the Lévy-Leblond theory for boson/fermion fields of arbitrary spin [21, 22], as well as its supersymmetric and anyon generalizations. Among particular examples, we consider the non-relativistic limit of a topologically massive vector gauge field.

In the second, “Jackiw-Nair” non-relativistic limit [23, 24], spin and light speed both tend simultaneously to infinity in a special way such that the (super) Poincaré symmetry is reduced to the exotic (super) Galilei symmetry [25, 26, 27, 28].

2 Wave equations

Our starting point is to posit the vector set of equations

$$V_\mu \psi = 0, \quad V_\mu = \beta P_\mu - i\epsilon_{\mu\nu\lambda} P^\nu \mathcal{J}^\lambda + m \mathcal{J}_\mu, \quad (2.1)$$

where $P_\mu = -i\partial_\mu$ and \mathcal{J}_μ generates (2+1)D Lorentz transformations, $[\mathcal{J}_\mu, \mathcal{J}_\nu] = -i\epsilon_{\mu\nu\lambda} \mathcal{J}^\lambda$. Such a system of wave equations was proposed before in [16] to describe anyons of mass $m > 0$ and spin β .

Contracting (2.1) with P^μ , \mathcal{J}^μ and $-i\epsilon^{\mu\nu\lambda} P_\nu \mathcal{J}_\lambda$ produces three equations, namely

$$(\beta P^2 + m P \mathcal{J}) \psi = 0, \quad (2.2)$$

$$((\beta - 1) P \mathcal{J} + m \mathcal{J}^2) \psi = 0, \quad (2.3)$$

$$(P^2 \mathcal{J}^2 + (P \mathcal{J})(m - P \mathcal{J})) \psi = 0. \quad (2.4)$$

If $\beta = 0$, these equations do not fix the value of the (2+1)D Poincaré Casimir operator P^2 . We suppose therefore that $\beta \neq 0$. Then the system (2.2)–(2.4) is equivalent to

$$P^2 (P^2 + m^2) \psi = 0, \quad (2.5)$$

$$(\beta P^2 + m P \mathcal{J}) \psi = 0, \quad (2.6)$$

$$(m^2 \mathcal{J}^2 + \beta(1 - \beta) P^2) \psi = 0. \quad (2.7)$$

It follows that the field ψ satisfies either $P^2 \psi = 0$, $P \mathcal{J} \psi = 0$, $\mathcal{J}^2 \psi = 0$, or

$$(P^2 + m^2) \psi = 0, \quad (2.8)$$

$$(P \mathcal{J} - \beta m) \psi = 0, \quad (2.9)$$

$$(\mathcal{J}^2 + \beta(\beta - 1)) \psi = 0. \quad (2.10)$$

The first case corresponds to the trivial representation of the (2+1)D Poincaré group with $P_\mu = 0$ and $\mathcal{J}_\mu = 0$ as seen in the frame $P^\mu = (p, p, 0)$. Excluding this case, we assume henceforth that ψ satisfies Eqns. (2.8)–(2.10) with $\mathcal{J}_\mu \neq 0$. Our vector system of equations implies, hence, the Klein-Gordon (2.8) and the Pauli-Lubanski condition (2.9). The consistency of our vector equations (2.1) then follows from

$$[V_\mu, V_\nu] = -i\epsilon_{\mu\nu\lambda} (m V^\lambda + P^\lambda (P \mathcal{J} - \beta m)). \quad (2.11)$$

When nontrivial solutions of (2.1) do exist, it follows that they describe an *irreducible* representation of the (2+1)D Poincaré group with nonzero mass m and spin $s = \beta$ ³.

Our vector equations determine the representation of the spin part of the Lorentz transformation; eq. (2.10) fixes the value of the Lorentz Casimir.

Taking into account Eq. (2.8) and passing to the rest frame $P^\mu = (\varepsilon m, 0, 0)$ where $\varepsilon = +1$ or -1 , we find that our covariant vector equations, (2.1), are equivalent to two independent equations, namely to

$$(\mathcal{J}_0 - \varepsilon \beta) \psi = 0, \quad (2.12)$$

$$(\mathcal{J}_1 - i\varepsilon \mathcal{J}_2) \psi = 0. \quad (2.13)$$

³The spin zero case can also be incorporated into the theory, see below.

From Eq. (2.13) we infer that for $\varepsilon = 1$, \mathcal{J}_μ should belong to a representation bounded from below, and for $\varepsilon = -1$ a representation has to be bounded from above ⁴. (It can also be bounded from both sides.)

The universal covering group of the (2+1)D Lorentz group admits half-bounded unitary infinite-dimensional representations, namely, the discrete type series D_α^\pm [12, 16]. They are characterized by a diagonal operator and a highest (lowest) vector,

$$D_\alpha^+ : \quad \mathcal{J}_0|\alpha, n\rangle = (\alpha + n)|\alpha, n\rangle \quad \text{and} \quad (\mathcal{J}_1 - i\mathcal{J}_2)|\alpha, 0\rangle = 0, \quad (2.14)$$

$$D_\alpha^- : \quad \mathcal{J}_0|\alpha, n\rangle = -(\alpha + n)|\alpha, n\rangle \quad \text{and} \quad (\mathcal{J}_1 + i\mathcal{J}_2)|\alpha, 0\rangle = 0, \quad (2.15)$$

where $n = 0, 1, \dots$. The representations D_α^+ and D_α^- are related via the Lorentz algebra automorphism $\mathcal{J}_0 \rightarrow -\mathcal{J}_0$, $\mathcal{J}_1 \rightarrow \mathcal{J}_1$, $\mathcal{J}_2 \rightarrow -\mathcal{J}_2$. The Casimir operator takes here the value,

$$\mathcal{J}^2 = -\alpha(\alpha - 1). \quad (2.16)$$

These unitary half-bounded infinite-dimensional representations D_α^\pm appear in most earlier works on wave equations for relativistic [13, 14, 15, 16, 27, 29, 30, 31, 32, 33, 34] and non-relativistic [22, 27] anyons.

The universal covering group of the (2+1)D Lorentz group admits also unbounded representations, namely, the principal and the supplementary continuous series. They only allow trivial solutions of the vector set of equations (2.1), and will therefore be discarded.

The non-unitary $(2j + 1)$ -dimensional representations \tilde{D}^j , $j = \frac{1}{2}, 1, \dots$, characterized by (2.16) with $\alpha = -j$ i.e., $\mathcal{J}^2 = -j(j + 1)$ ⁵, admits both highest- and lowest-spin states,

$$\tilde{D}^j : \quad \mathcal{J}_0|j, \ell\rangle = \ell|j, \ell\rangle, \quad (\mathcal{J}_1 - i\mathcal{J}_2)|j, 0\rangle = 0 \quad \text{and} \quad (\mathcal{J}_1 + i\mathcal{J}_2)|j, j\rangle = 0, \quad (2.17)$$

where $\ell = -j, -j + 1, \dots, j$. Hence, they also can be used in our equation. In this case, the vector equation describes usual bosons and fermions of arbitrary spin.

Up to an unitary transformation, they are obtained from the spin- j representations of $su(2)$, $\mathcal{J}_0 = \hat{\mathcal{J}}^3$, $\mathcal{J}_k = -i\hat{\mathcal{J}}^k$, $k = 1, 2$, where $[\hat{\mathcal{J}}^a, \hat{\mathcal{J}}^b] = i\epsilon^{abc}\hat{\mathcal{J}}^c$, $a, b, c = 1, 2, 3$.

There is a third type of representation of $\mathfrak{so}(2, 1)$ which plays an important role in our equations, namely *half-bounded, infinite-dimensional non-unitary representations*. These representations are characterized by relations of the form (2.16) and (2.14) or (2.15), with the parameter α in (2.16) taking *non-half-integer negative* values, $-\alpha \neq j = 1/2, 1, \dots$. They are denoted here by \tilde{D}_α^\pm ,

$$\tilde{D}_\alpha^\pm : \quad \mathcal{J}_0|\alpha, n\rangle = \pm(\alpha + n)|\alpha, n\rangle, \quad (\mathcal{J}_1 \mp i\mathcal{J}_2)|\alpha, 0\rangle = 0, \quad -j < \alpha < -j + 1/2. \quad (2.18)$$

These representations describe anyons whose fractional spin interpolates between bosons and fermions.

Occasionally, we will also use the unifying notation,

$$\mathcal{D}_\alpha^\pm = \begin{cases} D_\alpha^\pm & \text{for } \alpha > 0, \\ \tilde{D}_\alpha^\pm & \text{for } \alpha < 0, \alpha \neq -j, \\ \tilde{D}^j & \text{for } \alpha = -j. \end{cases} \quad (2.19)$$

⁴Changing $m \rightarrow -m$ in (2.1) together with intertwining representation bounded from below and above yields solutions of the same energy but of the opposite sign of spin.

⁵We reserve the notation j to denote the (half-) integer values, $1/2, 1, 3/2, \dots$

Choosing \mathcal{J}_μ in one of irreducible representations (2.19) and putting

$$\beta = \alpha, \quad (2.20)$$

Eq. (2.10) is identically satisfied for all real $\alpha \neq 0$.

Matrix representations of D_α^\pm , \tilde{D}^j and \tilde{D}_α^\pm are considered, in a unified way, in Section 3 below; in Section 6 they are treated as Fock-like representations of a certain deformation of the Heisenberg algebra.

In Section 4 we describe explicit solutions of the vector set of equations with \mathcal{J}_μ chosen in any of the three irreducible representations (2.19), and with parameter β coherently fixed by (2.20). In what follows we refer to such a realization of (2.1) as a *minimal* one.

An *extended* formulation, which combines several irreducible representations, will be considered in Section 5.

2.1 Examples: Dirac and Deser-Jackiw-Templeton fields

The two simplest finite-dimensional representations \tilde{D}^j are,

- $j = -\alpha = \frac{1}{2}$, $\mathcal{J}_\mu = -\frac{1}{2}\gamma_\mu$, when the linear differential equation (2.9) reduces to the Dirac equation,

$$(P\gamma - m)\psi = 0. \quad (2.21)$$

Conversely, our vector system (2.1) is the Dirac equation (2.21), multiplied by $\frac{1}{2}\gamma_\mu$.

- $j = -\alpha = 1$, $(\mathcal{J}_\mu)^\lambda{}_\nu = -i\epsilon^\lambda{}_{\mu\nu}$, when (2.9) reduces to the equation for a *topologically massive vector gauge field* [3],

$$\left(-i\epsilon^\lambda{}_{\mu\nu}P^\mu + m\delta^\lambda{}_\nu\right)\psi^\nu = 0, \quad (2.22)$$

where we switched to the notation ψ^μ instead of F^μ , used in (1.4). Eq. (2.22) implies the transversality condition

$$P_\mu\psi^\mu = 0, \quad (2.23)$$

that allows us to view the vector field ψ^μ as dual to the $U(1)$ gauge invariant tensor, $\psi_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda}F^{\nu\lambda}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Conversely, the vectorial form (2.1) is obtained from (2.22) by multiplication by $-i\epsilon^\rho{}_{\sigma\lambda}$ and addition to Eq. (2.23), multiplied by $-\delta^\rho{}_\sigma$.

The values $j = \frac{1}{2}$ and $j = 1$ are exceptional in that in these cases (only) the linear differential equation (2.9), i.e. (2.21) and (2.22), does already imply the Klein-Gordon equation, (2.8). This explains why, for $|s| \neq \frac{1}{2}, 1$, getting an irreducible representation of the Poincaré group requires positing a vector set of equations for anyons, and for usual fields of integer and half-integer spin with $|s| = j$, $j > 1$ ⁶.

⁶Any two of three equations (2.1) generate the third one as consistency condition, see Section 4 below; the complete set of three equations provides us with an explicitly covariant formulation.

2.2 Example: Majorana-Dirac anyon field

Both the Dirac and DJT models are associated with finite dimensional representations. We can also combine them with fractional spin, however. Consider, in fact, the representation of the spin part of the Lorentz generators as ⁷,

$$\mathcal{J}_\mu = J_\mu - \frac{1}{2}\gamma_\mu, \quad J_\mu \in D_\alpha^+, \quad -\frac{1}{2}\gamma_\mu \in \tilde{D}^{1/2}, \quad \alpha \neq 1/2, \quad (2.24)$$

and put

$$\beta = \alpha - 1/2. \quad (2.25)$$

Eq. (2.10) allows us to infer,

$$\Pi\psi = 0, \quad \text{where} \quad \Pi = J\gamma + \alpha. \quad (2.26)$$

The operator Π is a projector : $\Pi^2 = (2\alpha - 1)\Pi$. Multiplying (2.26) from the left by the [on-shell invertible, cf. (2.8)] operator $P\gamma$, we find that the field also has to satisfy

$$\Upsilon\psi = 0, \quad \text{where} \quad \Upsilon = PJ - \alpha P\gamma - \Lambda, \quad \Lambda = i\epsilon_{\mu\nu\lambda}P^\mu J^\nu \gamma^\lambda. \quad (2.27)$$

Multiplication of (2.26) by $-\Lambda$ yields $((\alpha - 1)\Upsilon - \Lambda)\psi = 0$, and we find, hence, that ψ also has to satisfy the equation

$$\Lambda\psi = 0. \quad (2.28)$$

In the present case, Eq. (2.9) reads $(PJ - \alpha m) - \frac{1}{2}(P\gamma - m)\psi = 0$. Combining it with Eq. (2.27) and taking into account Eq. (2.28), we find that our field ψ_b^n , that carries an index $n = 0, 1, \dots$ of the representation D_α^+ as well as a spinor index b , has to satisfy, separately, the $(2+1)D$ Majorana and Dirac equations,

$$(PJ - \alpha m)\psi = 0, \quad (P\gamma - m)\psi = 0. \quad (2.29)$$

This system of two equations was proposed in [14] to describe $(2+1)D$ anyons; the corresponding field was called a ‘‘Majorana-Dirac field’’.

It is easy to check that these two equations imply (2.26) and (2.28) as consistency (integrability) conditions. Moreover, we have the following remarkable property: the full set of four equations (2.29), (2.26) and (2.28) is generated by positing any two equations out of the four, (one of which should involve the mass parameter). It follows that the vector system (2.1), with (2.24) and (2.25) describes an irreducible representation of the $(2+1)D$ Poincaré group of mass m and spin $s = \alpha - \frac{1}{2} \neq 0$.

2.3 Example: Jackiw-Nair anyon field

Similarly, consider now,

$$\mathcal{J}_\mu = J_\mu + j_\mu, \quad J_\mu \in D_\alpha^+, \quad (j_\mu)^\lambda{}_\nu = -i\epsilon^\lambda{}_{\mu\nu} \in \tilde{D}^1, \quad \alpha \neq 1, \quad (2.30)$$

and

$$\beta = \alpha - 1, \quad (2.31)$$

for which the Pauli-Lubanski condition (2.9) becomes,

$$\left(P_\mu J^\mu \delta_\nu^\lambda - i\epsilon^\lambda{}_{\mu\nu} P^\mu - (\alpha - 1)m\delta_\nu^\lambda \right) \psi^\nu = 0, \quad (2.32)$$

⁷The assumed tensor product, $\mathcal{J}_\mu = J_\mu \otimes 1 - 1 \otimes \frac{1}{2}\gamma_\mu$, is not indicated for the sake of notational simplicity.

that is the anyon equation put forward by Jackiw and Nair [13]⁸. Eq. (2.10) reads in turn

$$\left(-i\epsilon^\lambda_{\mu\nu}J^\mu - \alpha\delta^\lambda_\nu\right)\psi^\nu = 0. \quad (2.33)$$

Now we turn to the subsidiary conditions in [13]. Contracting (2.33) from the left with J_λ yields

$$J_\mu\psi^\mu = 0. \quad (2.34)$$

Next, contraction with the vector P_λ produces $(-i\epsilon_{\mu\nu\lambda}P^\nu J^\lambda - \alpha P_\mu)\psi^\mu = 0$, while contraction with $-i\epsilon_{\lambda\rho\sigma}P^\rho J^\sigma$ produces an equation of the same form, but with the sign before α reversed. In addition to (2.34), the field also has to satisfy therefore,

$$P_\mu\psi^\mu = 0, \quad \text{and} \quad \epsilon_{\mu\nu\lambda}P^\mu J^\nu\psi^\lambda = 0, \quad (2.35)$$

which are precisely the subsidiary conditions of Jackiw and Nair in Ref. [13]⁹.

Contracting Eq. (2.33) with $i\epsilon^\rho_{\sigma\lambda}P^\sigma$ and taking into account Eq. (2.35), yields

$$\left((PJ - \alpha m)\delta^\lambda_\nu + \alpha(-i\epsilon^\lambda_{\mu\nu}P^\mu + m\delta^\lambda_\nu)\right)\psi^\nu = 0. \quad (2.36)$$

Then, using Eq. (2.32) shows that Eq. (2.33) splits the JN equation (2.32) into a Majorana equation, supplemented with the DJT equation of a topologically massive gauge vector field. Note however that, unlike in the Majorana-Dirac case (2.29), imposing the pair of Majorana and DJT equations alone on ψ_μ^n does not imply the condition (2.33) that guarantees the irreducibility equation (2.10) as well as the other necessary subsidiary conditions in (2.35).

We shall see in Section 5 that Majorana-Dirac and Jackiw-Nair descriptions of anyon fields are included as particular cases into a broader, *extended* realization of our basic set of equations (2.1).

3 $\mathfrak{so}(2, 1)$ lowest/highest weight representations

Now we discuss some aspects of the representation theory of $\mathfrak{so}(2, 1)$, which will be used in the next two Sections to derive explicit solutions. This will allow us to see, in particular, how the non-unitary infinite dimensional representations \tilde{D}_α^+ (or \tilde{D}_α^-) produce an anyon interpolation between bosons and fermions, and how bosons and fermions can be composed from anyons.

Let \mathcal{J}_μ generate the $\mathfrak{so}(2, 1)$ algebra, $[\mathcal{J}_\mu, \mathcal{J}_\nu] = -i\epsilon_{\mu\nu\lambda}\mathcal{J}^\lambda$. In terms of the ladder operators $\mathcal{J}_\pm = \mathcal{J}_1 \pm i\mathcal{J}_2$,

$$[\mathcal{J}_-, \mathcal{J}_+] = 2\mathcal{J}_0, \quad [\mathcal{J}_0, \mathcal{J}_\pm] = \pm\mathcal{J}_\pm. \quad (3.1)$$

Eq. (3.1) implies

$$[\mathcal{J}_-, \mathcal{J}_+^n] = 2n\mathcal{J}_+^{n-1}\left(\frac{1}{2}(n-1) + \mathcal{J}_0\right), \quad n = 0, 1, 2, \dots \quad (3.2)$$

From now on, for the sake of definiteness, we consider the lowest weight representations built over the state $|0\rangle$,

$$\mathcal{J}_0|0\rangle = \alpha|0\rangle, \quad \mathcal{J}_-|0\rangle = 0, \quad \langle 0|0\rangle = 1, \quad (3.3)$$

⁸Both in (2.32) and in (2.33) the index n of ψ_μ^n has been suppressed.

⁹The subsidiary condition (4.12a) of [13] is a linear combination of (2.34) and of the first equation from (2.35), $(P_\mu J_\nu - J_\mu P_\nu)\psi^\nu = 0$.

where α is real ¹⁰.

The non-normalized states

$$|\widetilde{n}\rangle = \mathcal{J}_+^n |0\rangle, \quad n = 0, 1, \dots, \quad (3.4)$$

are eigenvectors of \mathcal{J}_0 ,

$$\mathcal{J}_0 |\widetilde{n}\rangle = (\alpha + n) |\widetilde{n}\rangle, \quad (3.5)$$

on which the $\mathfrak{so}(2, 1)$ Casimir $\mathcal{C}_{\mathfrak{so}(2,1)} = -\mathcal{J}_0^2 + \frac{1}{2}(\mathcal{J}_+\mathcal{J}_- + \mathcal{J}_-\mathcal{J}_+)$ takes the value

$$\mathcal{C}_{\mathfrak{so}(2,1)} |\widetilde{n}\rangle = -\alpha(\alpha - 1) |\widetilde{n}\rangle. \quad (3.6)$$

From relations (3.2)–(3.4) it follows that the vectors $|\widetilde{n}\rangle$ form an orthogonal basis,

$$\langle \widetilde{n} | \widetilde{n'} \rangle = \delta_{nn'} \mathcal{C}_{\alpha,n}^2, \quad \mathcal{C}_{\alpha,0} = 1, \quad \mathcal{C}_{\alpha,n}^2 = n! \prod_{k=0}^{n-1} (2\alpha + k), \quad (3.7)$$

where $\mathcal{C}_{\alpha,n}^2$ is a squared norm of $|\widetilde{n}\rangle$.

For $\alpha \neq -j$, $j = 0, 1/2, 1, \dots$, the numbers $\mathcal{C}_{\alpha,n}^2$ take nonzero values for any $n = 0, 1, \dots$; the corresponding representations are infinite-dimensional and irreducible.

For $\alpha = -j$, only the first $2j + 1$ numbers $\mathcal{C}_{\alpha,n}^2$ with $n = 0, \dots, 2j$ take nonzero values, while $\mathcal{C}_{\alpha,n}^2 = 0$ for $n = 2j + 1, 2j + 2, \dots$. This happens if $|\widetilde{n}\rangle = 0$ for $n = 2j + 1, \dots$, and then we have a $(2j + 1)$ -dimensional irreducible representation.

There is also another possibility, though : the states $|\widetilde{n}\rangle$ with $n = 2j + 1, 2j + 2, \dots$ may be nonzero. Curiously, they have zero norm.

We first consider *irreducible* representations, where no zero norm states of the form (3.4) are assumed to appear, and then discuss how to treat representations with zero norm states.

3.1 Irreducible representations

To analyze the solutions of the wave equations, it is convenient to work with normalized states, denoted by $|n\rangle$,

$$|0\rangle = |\widetilde{0}\rangle, \quad (\mathcal{J}_+)^n |0\rangle = (\prod_{k=0}^{n-1} \mathcal{C}_k^\alpha) |n\rangle = |\widetilde{n}\rangle, \quad (3.8)$$

$$\mathcal{C}_n^\alpha = \sqrt{(2\alpha + n)(n + 1)}. \quad (3.9)$$

Note that $\mathcal{C}_{\alpha,n}^2 = \prod_{k=0}^{n-1} \mathcal{C}_k^\alpha$, cf. (3.7), and that

$$\mathcal{J}_0 |n\rangle = (\alpha + n) |n\rangle, \quad \mathcal{J}_+ |n\rangle = \mathcal{C}_n^\alpha |n + 1\rangle, \quad \mathcal{J}_- |n\rangle = \mathcal{C}_{n-1}^\alpha |n - 1\rangle. \quad (3.10)$$

Due to the structure of the squared norm $\mathcal{C}_{\alpha,n}^2$, the following cases have to be distinguished.

- For $\alpha = 0$, we have $\mathcal{C}_0^0 = 1$ [$\mathcal{C}_n^0 = 0$, $n = 1, \dots$], yielding the *trivial* one-dimensional spin-0 representation, $\mathcal{J}_\mu |0\rangle = 0$.
- D_α^+ where $\alpha > 0$. The coefficients $\mathcal{C}_{\alpha,n}^2$ are positive for any n . The $|n\rangle = (\mathcal{C}_{\alpha,n})^{-1} |\widetilde{n}\rangle$ are just those $|\alpha, n\rangle$ in (2.14), and satisfy the orthonormality relation $\langle n | n' \rangle = \delta_{nn'}$. This corresponds to the *infinite-dimensional unitary half-bounded representations*.

¹⁰Highest weight representations can be analyzed in the same way and can also be obtained directly from the lowest weight representations via the Lorentz algebra automorphism $\mathcal{J}_0 \rightarrow -\mathcal{J}_0$, $\mathcal{J}_\pm \rightarrow \mathcal{J}_\mp$.

- \tilde{D}^j with $\alpha = -j$, a negative (half)integer. We have both a lowest weight vector, $\mathcal{J}_-|0\rangle = 0$, and a highest weight vector, $\mathcal{J}_+|2j\rangle = 0$; the coefficient C_{2j}^{-j} in (3.10) vanishes. This is the *usual* $(2j+1)$ -dimensional non-unitary representation.

By (3.9), the coefficients C_n^{-j} , $n = 1, \dots, 2j-1$, are imaginary. Put

$$C_n^{-j} = iC_n'^{-j}, \quad C_n'^{-j} = \sqrt{(2j-n)(n+1)}. \quad (3.11)$$

Making use of the $\mathfrak{so}(2,1)$ commutation relations, one finds that the states $|n\rangle$, $n = 0, 1, \dots, 2j$, satisfy $\langle n|n'\rangle = (-1)^n \delta_{nn'}$. The corresponding metric in the vector space \tilde{D}^j ,

$$\eta_{nn'} = \text{diag}(1, -1, \dots, (-1)^{2j-1}, (-1)^{2j}), \quad (3.12)$$

is indefinite, consistently with the non-unitary character of such representations. Redefining the generators as $\mathcal{J}_\pm \rightarrow \pm i\mathcal{J}_\pm$ yields, instead of (3.10),

$$\mathcal{J}_+|n\rangle = C_n'^{-j}|n+1\rangle, \quad \mathcal{J}_-|n\rangle = -C_{n-1}'^{-j}|n-1\rangle, \quad (3.13)$$

where $C_n'^{-j}$ is given by (3.11). Such redefined operators \mathcal{J}_+ and \mathcal{J}_- are mutually conjugate, $(\Psi|\mathcal{J}_+\Phi)^* = (\Phi|\mathcal{J}_-\Psi)$, w.r.t. the indefinite scalar product $(\Psi, \Phi) = \bar{\Psi}_n \Phi^n$, $\bar{\Psi}_n = \Psi^{*k} \hat{\eta}_{kn}$, where $\Phi^n = \langle n|\Phi\rangle$, and matrix $\hat{\eta}$ is given by (3.12), while $(\Psi|\mathcal{J}_0\Phi)^* = (\Phi|\mathcal{J}_0\Psi)$.

- \tilde{D}_α^+ with α negative such that $\alpha \neq -j$. In this case the $\mathcal{C}_{\alpha,n}^2$ alternate. This corresponds to the *infinite-dimensional half-bounded non-unitary representations*.

Let us discuss this case in some detail. For

$$-j < \alpha < -j + \frac{1}{2}, \quad j = 1/2, 1, 3/2, \dots, \quad (3.14)$$

all coefficients (3.9) are nonzero. Those with index $n < 2j$ have an imaginary factor i , while those with $n \geq 2j$ are real. The metric is indefinite,

$$\eta_{nn'} = \text{diag}(1, -1, \dots, (-1)^{2j-1}, (-1)^{2j}, (-1)^{2j}, (-1)^{2j} \dots). \quad (3.15)$$

In the first $(2j+1)$ positions the metric alternates as in the finite-dimensional representation \tilde{D}^j , and after that it takes the same (constant) value $(-1)^{2j}$ up to infinity.

Both the finite and infinite non-unitary representations can be considered in a unified way by putting $\alpha = -(j+\epsilon)$ in (3.10). Redefining $\mathcal{J}_\pm \rightarrow \pm i\mathcal{J}_\pm$, we get

$$\mathcal{J}_0|n\rangle = (-j-\epsilon+n)|n\rangle, \quad \mathcal{J}_+|n\rangle = C_n'^{-(j+\epsilon)}|n+1\rangle, \quad \mathcal{J}_-|n\rangle = -C_{n-1}'^{-(j+\epsilon)}|n-1\rangle, \quad (3.16)$$

where,

$$C_n'^{-(j+\epsilon)} = \sqrt{(2(j+\epsilon)-n)(n+1)}. \quad (3.17)$$

Here the parameter $-1/2 \leq \epsilon \leq 0$ interpolates between $-1/2$ and 0 , which correspond to the finite dimensional representations $D^{j-1/2}$ and D^j , respectively.

In all the described cases, the structure of the metric is consistent with the properties of the $\mathcal{C}_{\alpha,n}^2$ in (3.7).

3.2 Representations with zero norm states

Consider now the *infinite-dimensional* representation with $\alpha = -j$, which we denote here by \widetilde{D}_{-j}^+ . The necessity to treat such peculiar representations comes from their appearance in the extended realization which will be considered in Section 5.

The representation \widetilde{D}_{-j}^+ is characterized by the presence of an infinite number of zero norm states of the form (3.4) with $n = 2j + 1 + k$, $k = 0, 1, \dots$. In such a representation, there is a peculiar *zero norm* state $\widetilde{|2j + 1\rangle} = \mathcal{J}_+^{2j+1}|0\rangle$, which by (3.2) and (3.5) satisfies

$$\mathcal{J}_- \widetilde{|2j + 1\rangle} = 0, \quad \mathcal{J}_0 \widetilde{|2j + 1\rangle} = (j + 1) \widetilde{|2j + 1\rangle}. \quad (3.18)$$

Due to (3.18), the infinite set of zero norm states $\widetilde{|2j + 1 + k\rangle} = \mathcal{J}_+^k \widetilde{|2j + 1\rangle}$, $k = 0, 1, \dots$, span a space invariant under the action of $\mathfrak{so}(2, 1)$, i.e., these states form an irreducible infinite-dimensional null subspace which we denote by \mathcal{D}_{j+1}^{+0} . All the infinite-dimensional representation \widetilde{D}_{-j}^+ spanned by the states (3.4) with $n = 0, 1, \dots$, is, therefore, reducible.

Notice that the invariant null subspace \mathcal{D}_{j+1}^{+0} may appear here because the value $-(-j)((-j) - 1) = -j^2 - j$ of the Casimir operator (3.6) at $\alpha = -j$ admits, by (3.18), also the alternative factorization, $-j^2 - j = -(j + 1)((j + 1) - 1)$.

Since the zero norm states $\widetilde{|2j + 1 + k\rangle}$, $k = 0, 1, \dots$ are orthogonal to all states $\widetilde{|n\rangle}$, $n = 0, 1, \dots$, we can work with equivalence classes, viewing $|\Psi\rangle$ and $|\Psi\rangle + |\varphi\rangle$ where $|\Psi\rangle \in \widetilde{D}_{-j}^+$ and $|\varphi\rangle \in \mathcal{D}_{j+1}^{+0}$ as equivalent¹¹. In particular, we consider any state $|\varphi\rangle$ equivalent to the zero state. In such a way, we reduce the peculiar infinite-dimensional representation \widetilde{D}_{-j}^+ to the $(2j + 1)$ -dimensional non-unitary irreducible representation \tilde{D}^j described above, i.e., we get $\widetilde{D}_{-j}^+ / \mathcal{D}_{j+1}^{+0} = \tilde{D}^j$.

4 Solutions of the wave equations. Anyon interpolation between bosons and fermions

Having in mind the three essentially different types of irreducible representations, we analyze first the solutions of our vector system in the minimal realization $\beta = \alpha$, assuming that the wave function ψ carries irreducible representation \mathcal{D}_α^+ , whose type is defined by the value of the parameter α .

The wave function in (2.1) can be decomposed into partial waves,

$$\psi(x) = \sum_{n=0}^r \psi_n(x) |n\rangle, \quad (4.1)$$

where $r = \infty$ for the representations D_α^+ and \tilde{D}_α^+ , and $r = 2j$ for the finite-dimensional representation \tilde{D}^j ($\alpha = -j$).

Having chosen an irreducible representation, (2.10) is satisfied identically. The three equations in (2.1) are not independent: choosing any two of them, the third one follows as a consequence.

¹¹Mathematicians call \widetilde{D}_{-j}^+ a “Verma module”.

It is convenient to choose equations (2.1) with $\mu = 1, 2$. Their complex linear combinations $V_{\pm}\psi = 0$, $V_{\pm} = V_1 \pm iV_2$, give then the system of two equations

$$[(\alpha - \mathcal{J}_0)P_+ + (m + P_0)\mathcal{J}_+] \psi = 0, \quad [(\alpha + \mathcal{J}_0)P_- + (m - P_0)\mathcal{J}_-] \psi = 0, \quad (4.2)$$

where $P_{\pm} = P_1 \pm iP_2$. Consistently with (2.11), these two equations generate the one with $\mu = 0$ as an integrability condition,

$$\left[\alpha P_0 + m\mathcal{J}_0 + \frac{1}{2}(P_-\mathcal{J}_+ - P_+\mathcal{J}_-) \right] \psi = 0. \quad (4.3)$$

With (4.1), (3.9) and (3.10), from (4.2) we get the equivalent system

$$\sqrt{n+2\alpha}(m - P^0)\psi_n - \sqrt{n+1}P_+\psi_{n+1} = 0, \quad (4.4)$$

$$\sqrt{n+2\alpha}P_-\psi_n + \sqrt{n+1}(m + P^0)\psi_{n+1} = 0. \quad (4.5)$$

Then some algebraic manipulations yield the component form of Eq. (4.3),

$$(\alpha P_0 + m(\alpha + n))\psi_n + \frac{1}{2}\left(\sqrt{(2\alpha + n - 1)n}P_-\psi_{n-1} - \sqrt{(2\alpha + n)(n+1)}P_+\psi_{n+1}\right) = 0. \quad (4.6)$$

Eqns (4.4), (4.5) are conveniently analyzed in the momentum representation. For positive energy, $P^0 > 0$, the operator $P^0 + m \neq 0$ is invertible. Then (4.5) allows us to write all partial wave components ψ_n in terms of ψ_0 . From (4.5) one obtains, first,

$$\psi_{n+1} = -\frac{\sqrt{n+2\alpha}}{\sqrt{n+1}} \frac{P_-}{P^0 + m} \psi_n, \quad P^0 > 0, \quad (4.7)$$

and then iteratively,

$$\psi_n = \sqrt{\mathcal{C}_n} \left(\frac{-P_-}{P^0 + m} \right)^n \psi_0, \quad (4.8)$$

$$\mathcal{C}_n = \frac{2\alpha(2\alpha+1)\dots(2\alpha+n-1)}{n!} = (nB(2\alpha, n))^{-1}, \quad (4.9)$$

where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is Euler's beta function.

Note for further reference that if 2α is a negative integer, then the coefficients \mathcal{C}_n vanish when $n \geq 1 - 2\alpha$: the infinite tower of the ψ 's terminates, and the representation space becomes *finite dimensional*. Alternatively, the beta function in (4.9) has poles at negative integers.

The mass-shell condition follows from Eqns. (4.2), and so the component ψ_0 has the form

$$\psi_0 = C \delta(P^0 - \sqrt{\vec{p}^2 + m^2}) \delta(\vec{P} - \vec{p}), \quad (4.10)$$

where C is a (normalization) constant and \vec{p} is the momentum of the state.

Similarly, for negative energy, $P^0 - m$ is invertible, allowing us to express ψ_n in terms of ψ_{n+1} ,

$$\psi_n = \frac{\sqrt{n+1}}{\sqrt{n+2\alpha}} \frac{P_+}{m - P^0} \psi_{n+1}, \quad P^0 < 0. \quad (4.11)$$

All components can be expressed in terms of the highest spin state component. The highest spin state exists, however, *only* for the finite dimensional representations \tilde{D}^j , in which case $\beta = \alpha = -j$, and we get

$$\psi_{2j-n} = i^{-n} \sqrt{\frac{2j(2j-1)\dots(2j-n+1)}{n!}} \left(\frac{P_+}{P^0 - m} \right)^n \psi_{2j}, \quad (4.12)$$

where

$$\psi_{2j} = C \delta(P^0 + \sqrt{\vec{p}^2 + m^2}) \delta(\vec{P} - \vec{p}). \quad (4.13)$$

Let us now analyze the positive-energy solutions given by Eqns. (4.8), (4.9), (4.10). In the rest frame $\vec{p} = 0$, only the lowest component ψ_0 is nontrivial. For $\vec{p} \neq 0$, every subsequent component includes an additional kinematical factor $P_-/(P^0 + m)$.

For the infinite dimensional representations D_α^+ (and in the generic case of non-unitary representations \tilde{D}_α^+), the numerical coefficient \mathcal{C}_n is of order 1 when n tends to infinity.

However, in the case of non-unitary representations \tilde{D}_α^+ with α close to a negative (half)integer j , the nature of coefficients (4.9) is essentially different. When α , supposed to satisfy the relation (3.14), tends to $-j$ from above, the first coefficients with $n = 1, \dots, 2j$ tend to the values of those that correspond to the finite dimensional case \tilde{D}^j . In particular, $\mathcal{C}_{2j} \sim j + \alpha \rightarrow 0$ when $\alpha \rightarrow -j$. In such a case, all higher components are suppressed by the same factor $(j + \alpha)^{1/2} \rightarrow 0$.

With this picture, having also in mind the metric described in Section 3.1, we conclude that *the infinite-dimensional half-bounded non-unitary representations \tilde{D}_α^+ interpolate between boson and fermion states* of positive energy. This is what corresponds, intuitively, to an *anyon*.

It is worth noting that when α is between $-j$ and $-j + \frac{1}{2}$, $j = 1/2, 1, 3/2$ cf. (3.14) and approaches $-j + \frac{1}{2}$ from below, the structure with Poincaré spin $s = -j + \frac{1}{2}$ is recovered : it is described by the finite-dimensional non-unitary representation $\tilde{D}^{j-1/2}$ and by the indefinite metric (3.12), with j changed into $j - 1/2$.

The spin zero case $\beta = \alpha = 0$, excluded so far, can be incorporated as the limit $\alpha \rightarrow 0$ of $\mathcal{J}_\mu \in \tilde{D}_\alpha$, $\beta = \alpha < 0$.

5 Extended formulation. Bosons and fermions from anyons

In the general case \mathcal{J}_μ can be taken as the sum of the generators of the $(2+1)D$ Lorentz group in representations that correspond to the series D_α^\pm , \tilde{D}_α^\pm , and to \tilde{D}^j . The examples of Majorana-Dirac and Jackiw-Nair anyon fields considered in Sections 2.2 and 2.3 are just particular cases of such an *extended* formulation (or, realization) of the vector set of equations.

Taking, for simplicity, just two representations,

$$\mathcal{J}^\mu = J_\mu + J'_\mu, \quad J^2 = -\alpha(\alpha - 1), \quad J'^2 = -\alpha'(\alpha' - 1), \quad (5.1)$$

and put

$$\beta = \alpha + \alpha' \quad (5.2)$$

where, for a finite-dimensional representation, the parameter α and/or α' is $-j$, and (5.2) such that $\beta \neq 0$. Then Eq. (2.10) transforms into

$$(JJ' + \alpha\alpha') \psi = 0. \quad (5.3)$$

Eq. (5.3) is a non-dynamical, subsidiary equation for the field ψ [which carries two indices not shown explicitly]. It, in turn, implies that our composite system (5.1), (5.2) carries an *irreducible* representation of the $\mathfrak{so}(2,1)$ -spin $\beta = \alpha + \alpha'$, consistently with Eq. (2.10). This means that the extended formulation, in fact, recovers the previously considered minimal realization. It provides us, however, with a new possibility: in this formulation usual *boson and fermion fields of arbitrary spin can be composed from anyons*.

Before considering concrete examples, some general comments are in order.

• The operator of the non-dynamical equation (5.3) commutes with the total vector spin generator \mathcal{J}_μ . Solutions of (5.3) then can be given by the eingestates $\widetilde{|k\rangle}$ of the compact $\mathfrak{so}(2, 1)$ generator $\mathcal{J}_0 = J_0 + J'_0$, $\mathcal{J}_0 \widetilde{|k\rangle} = (\beta + k) \widetilde{|k\rangle}$, $k = 0, \dots$, presented as a certain linear combinations of the simultaneous J_0 and J'_0 eigenstates, $|n\rangle|n'\rangle$. Particularly, the state $|0\rangle = |0\rangle|0'\rangle$ is a solution that satisfies relations of the form (3.3) with α changed into $\beta = \alpha + \alpha'$ [for the sake of definiteness we assume here that $J_\mu \in \mathcal{D}_\alpha^+$, $J'_\mu \in \mathcal{D}_{\alpha'}^+$]. \mathcal{J}_μ commutes with the scalar operator $JJ' + \alpha\alpha' = -J_0J_0 + \frac{1}{2}(J_+J'_- + J_-J'_+) + \alpha\alpha'$; all other solutions $\widetilde{|k\rangle}$ of Eq. (5.3) are obtained therefore by subsequent application of \mathcal{J}_+ to the lowest weight state $|0\rangle$ cf. Section 3.

For $J_\mu \in \tilde{D}^j$ and $J'_\mu \in \tilde{D}^{j'}$, Eq. (5.3) singles out irreducible finite dimensional representation $\tilde{D}^{j+j'}$.

For $\beta \neq -j$, Eq. (5.3) separates an irreducible bounded from below infinite dimensional representation of the unitary or non-unitary type depending on $\beta > 0$ or $\beta < 0$.

When $\beta = -j$ and at least one of the $\mathfrak{so}(2, 1)$ spins is in an infinite dimensional representation, Eq. (5.3) gives a reducible representation \widetilde{D}_{-j}^+ with invariant infinite dimensional null subspace \mathcal{D}_{j+1}^{+0} . As it is explained in Section 3.2, working modulo zero norm states $|\varphi\rangle \in \mathcal{D}_{j+1}^{+0}$, solutions of (5.3) gives rise to the irreducible finite-dimensional non-unitary representation $\tilde{D}^j = \widetilde{D}_{-j}^+ / \mathcal{D}_{j+1}^{+0}$.

As a result, the solution of the vector system of equations in the extended formulation, for all possible choices of the $\mathfrak{so}(2, 1)$ representations in (5.1) is reduced to the minimal realization¹².

• When \mathcal{J}_μ is composed as in (5.1), the vector set of equations (2.1) splits, for each Lorentz index, into two vector sets.

Consider, for instance, $J_\mu \in D_\alpha^+$, $J'_\mu \in \tilde{D}^j$, assuming, as in the Majorana-Dirac and Jackiw-Nair cases, (2.24) and (2.30), that $\beta \neq 0$. In the rest frame the system (2.12) – (2.13) becomes

$$((J_0 - \varepsilon\alpha) + (J'_0 + \varepsilon j)) \psi = 0, \quad (5.4)$$

$$((J_1 - i\varepsilon J_2) + (J'_1 - i\varepsilon J'_2)) \psi = 0. \quad (5.5)$$

Eq. (5.5) has nontrivial solutions if and only if $(J_1 - i\varepsilon J_2) \psi = 0$. Such a solution, proportional to the lowest state $|\alpha, 0\rangle$ for the representation D_α^+ , is nontrivial only for $\varepsilon = +1$. Then Eq. (5.5) splits into

$$(J_1 - iJ_2) \psi = 0, \quad \text{and} \quad (J'_1 - iJ'_2) \psi = 0,$$

splitting Eq. (5.4) into

$$(J_0 - \alpha) \psi = 0, \quad \text{and} \quad (J'_0 + j) \psi = 0.$$

Then one finds that a solution of (2.1) has to be a simultaneous solution of two sets of equations of the form (2.1), in one of which \mathcal{J}_μ is $J_\mu \in D_\alpha^+$, and in the other \mathcal{J}_μ is $J'_\mu \in \tilde{D}^j$. The corresponding solution has positive energy, mass m and spin $s = \alpha - j \neq 0$. This means that the system of equations,

$$V_\mu^{(\alpha)} \psi = 0, \quad V_\mu^{\alpha'} \psi = 0, \quad JJ' + \alpha\alpha' = 0, \quad (5.6)$$

is equivalent to (2.1) with $\beta = \alpha + \alpha'$. In other words, spins simply add. This additive property allows us to create particles of spin β built from one of spin α and another one of spin α' . Intuitively, the non-dynamical constraint (5.3) “entangles” the component systems.

¹²The spin zero case $s = 0$ is incorporated into the extended formulation via a limit $\beta \rightarrow 0$ if representations in (5.1) are chosen in such a way that $-1/2 < \beta < 0$.

In the Majorana-Dirac and Jackiw-Nair examples discussed in Section 2, the entangling equation (5.3) takes the form (2.26) and (2.33), respectively.

- Choosing J_μ in a unitary infinite-dimensional representation D_α^+ with non-(half)integer α , and J'_μ in a finite-dimensional non-unitary representation \tilde{D}^j in such a way that $\beta = \alpha - j < 0$, the resulting irreducible representation is an infinite-dimensional half-bounded *non-unitary* representation of $\mathfrak{so}(2, 1)$. Such infinite-dimensional non-unitary representations belong to the Majorana-Dirac and Jackiw-Nair descriptions, no attention was paid to the corresponding interpolating anyons, in the original References [13, 14] though.

- The choices $J_\mu \in \tilde{D}_\alpha^+$, $J'_\mu \in \tilde{D}_{\alpha'}^+$ or $J'_\mu \in D_{\alpha'}^+$, provide us with another interesting case : if the parameters α and α' add up to a negative (half)-integer $\beta = -j$, our extended formulation, based on two half-bounded infinite-dimensional (i.e. anyon-like) representations, describes a usual relativistic finite-component field of spin j . In particular, from two anyon-like representations given by $\alpha = \alpha' = -1/4$ [considered below], the extended formulation reproduces the theory of the Dirac particle with spin $s = -1/2$.

Examples

Consider some representative particular examples.

- Let $J_\mu, J'_\mu \in \tilde{D}^{1/2}$, $\alpha = \alpha' = -1/2$, $\beta = -1$. The solutions of Eq. (5.3) are

$$|0\rangle = |0\rangle|0'\rangle, \quad |1\rangle = \frac{1}{\sqrt{2}} [|0\rangle|1'\rangle + |1\rangle|0'\rangle], \quad |2\rangle = |1\rangle|1'\rangle. \quad (5.7)$$

They are eigenstates of \mathcal{J}_0 with eigenvalues $-1, 0$ and $+1$, respectively, and satisfy the relations

$$\mathcal{J}_-|0\rangle = \mathcal{J}_+|2\rangle = 0, \quad \mathcal{J}_+|0\rangle = i\sqrt{2}|1\rangle, \quad \mathcal{J}_+|1\rangle = i\sqrt{2}|2\rangle, \quad (5.8)$$

$$(n|n') = (-1)^n \delta_{nn'}, \quad n, n' = 0, 1, 2, \quad (5.9)$$

[where we have used Eq. (3.9)] which correspond to the representation \tilde{D}^1 . The vector set of equations (2.1) describes, in this case, a topologically massive vector gauge field of mass m and spin $s = -1$, which can be treated as composed of two massive particles of spin $s = -1/2$ each.

- Let us take $(J_\mu)^\lambda{}_\nu = -i\epsilon^\lambda{}_{\mu\nu} \in \tilde{D}^1$, $\alpha = -1$, $J'_\mu = -\frac{1}{2}\gamma_\mu \in \tilde{D}^{1/2}$, $\alpha' = -1/2$, and $\beta = -3/2$. This corresponds to the Rarita-Schwinger field of spin $s = -3/2$ ¹³. Having in mind the splitting of our vector equations, we find that the vector-spinor field ψ^μ [spinor indices are not display explicitly] satisfies the Dirac and DJT equations, supplemented with the entangling equation (5.3),

$$(P\gamma - m)\psi^\mu = 0, \quad (5.10)$$

$$(-i\epsilon^\mu{}_{\nu\lambda}P^\nu + m\delta^\mu_\lambda)\psi^\lambda = 0, \quad (5.11)$$

$$(i\epsilon^\mu{}_{\nu\lambda}\gamma^\nu + \delta^\mu_\lambda)\psi^\lambda = 0. \quad (5.12)$$

Taking into account the identity $\gamma_\mu\gamma_\nu = -\eta_{\mu\nu} + i\epsilon_{\mu\nu\lambda}\gamma^\lambda$, simple algebraic manipulations show that the system (5.10)–(5.12) is equivalent to the Rarita-Schwinger equations

$$(P\gamma - m)\psi^\mu = 0, \quad \gamma_\mu\psi^\mu = 0. \quad (5.13)$$

¹³The chosen representation for $\alpha = -1$ differs from the matrix representation of Section 3 by a unitary transformation, see Eqns. (A.14), (A.17) and (A.16) in Appendix A.2.

Thus the system (5.13) generates, in itself, the DJT, and also the entangling, (5.12), equations. Notice a similarity between the second equation in (5.13) and Eq. (2.34).

• Consider now the case $J_\mu \in \tilde{D}_{-1/4}^+$, $J'_\mu \in \tilde{D}_{-1/4}^+$, $\alpha = \alpha' = -1/4$, $\beta = -1/2$. The two lowest normalized eigenstates of \mathcal{J}_0 are given formally by the same relations as in (5.7). It should be remembered, though, that the states which appear on the right hand sides of the relations belong to two copies of the infinite dimensional representation $\tilde{D}_{-1/4}^+$. The normalization is given by the indefinite metric described in Section 3. These states $|0\rangle$ and $|1\rangle$ have \mathcal{J}_0 -eigenvalues $-1/2$ and $+1/2$, respectively, and satisfy the relations

$$(n|n') = (-1)^n \delta_{nn'}, \quad n, n' = 1, 2, \quad \mathcal{J}_+|0\rangle = i|1\rangle. \quad (5.14)$$

Application of the ladder operator \mathcal{J}_+ to $|1\rangle$ does not annihilate it, but produces a zero-norm state,

$$\mathcal{J}_+|1\rangle = \frac{1}{\sqrt{2}} [|0\rangle|2'\rangle + |2\rangle|0'\rangle] + i|1\rangle|1'\rangle = |\widetilde{2}\rangle, \quad \langle\widetilde{2}|\widetilde{2}\rangle = 0,$$

which is orthogonal to $|0\rangle$ and $|1\rangle$, and is annihilated by \mathcal{J}_- .

Other solutions of Eq. (5.3) are obtained by applying, subsequently, \mathcal{J}_+ to the state $|\widetilde{2}\rangle$, which, together with the latter, span the null space $\mathcal{D}_{1/2}^{+0}$. Factoring out the states $|\varphi\rangle \in \mathcal{D}_{1/2}^{+0}$, we get the two-dimensional non-unitary representation $\tilde{D}^{1/2}$. In such a way two anyons of identical $\mathfrak{so}(2,1)$ spin $-1/4$, yield, in our extended realization, a fermion field of spin $-1/2$. By this reason, in our picture, these anyons *can* be called *semions*.

The picture we have just described can be reinterpreted in the following way. Consider a pair of positive energy states (particles) with Poincaré spins $-1/4$. Separately, each such particle can be described within the minimal realization with $J_\mu \in \tilde{D}_{-1/4}^+$, $\beta = \alpha = -1/4$. The wave function of each particle picks up a phase, $e^{-i\pi/2} = -i$, under a 2π spatial rotation generated by J_0 , and the system can be interpreted as a pair of *semions*.

When switching to the extended formulation with $J_\mu \in \tilde{D}_{-1/4}^+$, $J'_\mu \in \tilde{D}_{-1/4}^+$, $\alpha = \alpha' = -1/4$, $\beta = -1/2$, the vector system of equations splits, as discussed above, into two vector systems of equations, each describing a semion field. These two fields are not more independent, however. The vector system implies the non-dynamical equation (5.3) that guarantees the irreducibility of the Poincaré representation realized on our composed, two-semion system. This equation introduces a kind of entanglement between the two semion states, that provides us, as a result, with a spin $s = -1/2$ fermion. Each independent semion is described by an infinite-component field in a frame different from the rest frame [with higher components suppressed by a kinematical factor, see Eq. (4.8)]. Equation (5.3) then introduces a kind of destructive interference between the semion wave functions, with an infinite number of components associated with the null subspace, with the exception of two surviving independent wave components, which, due to the entanglement, describe a two-component positive-energy fermion state.

Notice here that if we choose instead two unitary infinite dimensional representations, $J_\mu \in D_{+1/4}^+$, $J'_\mu \in D_{+1/4}^+$, $\alpha = \alpha' = +1/4$, $\beta = +1/2$, our vector set of wave equations would describe a positive-energy particle of mass m and spin $+1/2$. Although its wave function picks up a phase -1 under a 2π rotation, we do not get a usual fermion, since its wave function has infinite number of components when $\vec{p} \neq 0$, cf. the previous Section. Anyons described by the unitary representations $D_{+1/4}^+$ can not be referred to, therefore, as semions¹⁴.

¹⁴The name ‘semions’ cannot be applied therefore to the supersymmetric system constructed in [30], based on

In the same way, if we add two spins corresponding to infinite dimensional anyon representations such that $\alpha + \alpha' = \beta = -j$, we can treat the resulting boson or fermion field as composed of two anyons.

• Finally, we consider shortly another particular interesting case, $J_\mu \in D_{1/2}^+$, $J'_\mu \in \tilde{D}^1$, $\alpha = 1/2$, $\alpha' = -1$, $\beta = -1/2$. In our framework, it describes a usual fermion field¹⁵.

According to Eq. (3.9), the nonzero coefficients are $C_0^{-1} = C_1^{-1} = i\sqrt{2}$, and $C_n^{1/2} = n + 1$. The $-1/2$ and $+1/2$ eigenstates of \mathcal{J}_0 are

$$|0\rangle = |0\rangle|0'\rangle \quad \text{and} \quad |1\rangle = \mathcal{J}_+|0\rangle = |1\rangle|0'\rangle + i\sqrt{2}|0\rangle|1'\rangle, \quad (5.15)$$

which satisfy the relations (5.14). The lowest zero-norm state is given by

$$\mathcal{J}_+|1\rangle = |\widetilde{2}\rangle \quad \text{where} \quad |\widetilde{2}\rangle = 2 \left[|2\rangle|0'\rangle + i\sqrt{2}|1\rangle|1'\rangle - |0\rangle|2'\rangle \right]. \quad (5.16)$$

$\mathcal{J}_-|\widetilde{2}\rangle = 0$. Higher zero norm states of the null subspace $\mathcal{D}_{1/2}^{+0}$ are produced by applying the ladder operator \mathcal{J}_+ to $|\widetilde{2}\rangle$.

6 Deformed oscillator representation of $\mathfrak{osp}(1|2)$

To get a concrete realization of the supersymmetric extensions of our model we shall discuss later, we use the reflection-deformed Heisenberg algebra (RDHA). The latter was (implicitly) introduced by Wigner [17] and led subsequently to parastatistics [18], that condensed, finally, into QCD color [19]. To make our discussion self-contained, we outline below those representations of the RDHA which are necessary for our purposes. See Ref. [20]¹⁶ for more details.

Let us consider the reflection-deformed Heisenberg algebra of the harmonic oscillator

$$[a^-, a^+] = 1 + \nu R, \quad R^2 = 1, \quad \{a^\pm, R\} = 0, \quad (6.1)$$

where ν is a real deformation parameter and R is the reflection operator, $R = (-1)^{\mathcal{N}} = \exp(i\pi\mathcal{N})$. The number operator

$$\mathcal{N} = \frac{1}{2}\{a^+, a^-\} - \frac{1}{2}(\nu + 1), \quad [\mathcal{N}, a^\pm] = \pm a^\pm, \quad (6.2)$$

defines the Fock space,

$$\mathcal{F} = \{|n\rangle, n = 0, 1, 2, \dots\}, \quad \mathcal{N}|n\rangle = n|n\rangle, \quad (6.3)$$

where $|n\rangle = C_n(a^+)^n|0\rangle$, $n = 0, 1, \dots$, $a^-|0\rangle = 0$, $\langle 0|0\rangle = 1$,

$$C_n = ([n]_\nu!)^{-1/2}, \quad [0]_\nu! = 1, \quad [n]_\nu! = \prod_{l=1}^n [l]_\nu, \quad n \geq 1, \quad [l]_\nu = l + \frac{1}{2}(1 - (-1)^l)\nu. \quad (6.4)$$

The explicit form of the coefficients (6.4) indicates that for $\nu > -1$ the algebra has infinite-dimensional, parabosonic-type unitary irreducible representations; for negative-odd values

the two infinite dimensional representations $D_{+1/4}^+$ and $D_{+3/4}^+$; the term ‘quartions’ used alternatively seems to be more appropriate.

¹⁵This case formally corresponds to the Jackiw-Nair scheme, no attention was paid to it in [13], though.

¹⁶In the context of quantum mechanical supersymmetry this algebra was used in [32, 35].

$\nu = -(2r + 1)$, $r = 1, 2, \dots$, it has $(2r + 1)$ -dimensional, non-unitary parafermionic-type representations, see below¹⁷. For $\nu < -1$, $\nu \neq -(2r + 1)$, it has infinite dimensional non-unitary representations of the \tilde{D}_α^\pm type.

The quadratic operators

$$\mathcal{J}_0 = \frac{1}{4}\{a^+, a^-\}, \quad \mathcal{J}_\pm = \mathcal{J}_1 \pm i\mathcal{J}_2 = \frac{1}{2}(a^\pm)^2, \quad (6.5)$$

together with linear operators

$$\mathcal{L}_a = \left(\frac{a^+ + a^-}{\sqrt{2}}, i \frac{a^+ - a^-}{\sqrt{2}} \right), \quad a = 1, 2, \quad (6.6)$$

generate the $\mathfrak{osp}(1|2)$ superalgebra,

$$[\mathcal{J}_\mu, \mathcal{J}_\nu] = -i\epsilon_{\mu\nu\lambda}J^\lambda, \quad (6.7)$$

$$\{\mathcal{L}_a, \mathcal{L}_b\} = 4i(\mathcal{J}\gamma)_{ab}, \quad [\mathcal{J}_\mu, \mathcal{L}_a] = \frac{1}{2}(\gamma_\mu)_a{}^b \mathcal{L}_b. \quad (6.8)$$

The Lorentz operators \mathcal{J}_μ are even, and \mathcal{L}_a are odd supercharges with respect to a \mathbb{Z}_2 -grading provided by the reflection operator, $[R, \mathcal{J}_\mu] = 0$, $\{R, \mathcal{L}_a\} = 0$. The $(2 + 1)$ dimensional Dirac matrices are in the Majorana representation,

$$(\gamma^0)_a{}^b = -(\sigma_2)_a{}^b, \quad (\gamma^1)_a{}^b = i(\sigma_1)_a{}^b, \quad (\gamma^2)_a{}^b = i(\sigma_3)_a{}^b, \quad (6.9)$$

which satisfy,

$$(\gamma_\mu)_a{}^\rho (\gamma_\nu)_\rho{}^b = -\eta_{\mu\nu} \epsilon_a{}^b + i\epsilon_{\mu\nu\lambda} (\gamma^\lambda)_a{}^b, \quad \gamma_{ab}^\mu = \gamma_{ba}^\mu, \quad \gamma_{ab}^{\mu\dagger} = -\gamma_{ab}^\mu. \quad (6.10)$$

The antisymmetric tensor, $\epsilon^{ab} = -\epsilon^{ba}$ ($\epsilon^{12} = 1$), plays the role of a spinor metric, rising and lowering indices, $A^a = \epsilon^{ab}A_b$, $A_a = A^b\epsilon_{ba}$. The scalar product has the property $A^a B_a = -A_a B^a$ for any spinors A_a and B_a . Then the Lorentz generators can be written as

$$\mathcal{J}_\mu = \frac{i}{4}\mathcal{L}^a (\gamma_\mu)_a{}^b \mathcal{L}_b, \quad \mu = 0, 1, 2. \quad (6.11)$$

The representation of $\mathfrak{osp}(1|2)$ built in this way is irreducible, characterized by the super-Casimir operator

$$\mathcal{C} = \mathcal{J}_\mu \mathcal{J}^\mu - \frac{i}{8}\mathcal{L}^a \mathcal{L}_a = \frac{1}{16}(1 - \nu^2). \quad (6.12)$$

The representation of the Lorentz subalgebra is, however, reducible, due to supersymmetry. This is reflected by the eigenvalues of the Casimir of the Lorentz subalgebra,

$$\mathcal{J}_\mu \mathcal{J}^\mu = -\hat{\alpha}(\hat{\alpha} - 1) \quad \text{with} \quad \hat{\alpha} = \frac{1}{4}(2 + \nu) - \frac{1}{4}R. \quad (6.13)$$

The irreducible components are obtained by projecting to the eigenspaces of R ,

$$R\mathcal{F}_\pm = \pm\mathcal{F}_\pm, \quad (6.14)$$

$$\mathcal{F}_+ = \{|n\rangle_+ = |2n\rangle, n = 0, 1, 2, \dots\}, \quad \mathcal{F}_- = \{|n\rangle_- = |2n + 1\rangle, n = 0, 1, 2, \dots\}. \quad (6.15)$$

¹⁷The Fock space construction here is similar to that in Section 3. We refrain from repeating similar technical details related to the appearance of null subspaces at $\nu = -(2r + 1)$.

On these subspaces, the $\mathfrak{so}(2, 1)$ Casimir (6.13) takes the values

$$\mathcal{J}_\mu \mathcal{J}^\mu \mathcal{F}_\pm = -\alpha_\pm(\alpha_\pm - 1)\mathcal{F}_\pm, \quad \text{where} \quad \alpha_+ = \frac{1+\nu}{4}, \quad \alpha_- = \alpha_+ + \frac{1}{2}. \quad (6.16)$$

The irreducible representations of the Lorentz algebra are extracted by the projectors, $\mathcal{F}_\pm = \Pi_\pm \mathcal{F}$,

$$J_\mu^{(\pm)} = \mathcal{J}_\mu \Pi_\pm, \quad [J_\mu^{(\pm)}, J_\nu^{(\pm)}] = -i\epsilon_{\mu\nu}^\lambda J_\lambda^{(\pm)}, \quad [J_\mu^{(+)}, J_\nu^{(-)}] = 0, \quad (6.17)$$

$$J_\mu^{(\pm)} J^{(\pm)\mu} = -\alpha_\pm(\alpha_\pm - 1)\Pi_\pm, \quad (6.18)$$

$$\Pi_\pm = \frac{1}{2}(1 \pm R), \quad \Pi_\pm^2 = \Pi_\pm, \quad \Pi_+ \Pi_- = 0, \quad \Pi_+ + \Pi_- = 1. \quad (6.19)$$

The representation of \mathcal{J}_μ in (6.5) is therefore a direct sum, $\mathcal{J}_\mu = J_\mu^{(+)} + J_\mu^{(-)}$. Consistently with (6.5) and (6.2), \mathcal{J}_0 has eigenvalues $n + \alpha_\pm$,

$$\mathcal{J}_0 |n\rangle_\pm = (n + \alpha_\pm) |n\rangle_\pm, \quad |n\rangle_\pm \in \mathcal{F}_\pm. \quad (6.20)$$

For future use, it is convenient to introduce the shifted number operator,

$$\mathbf{N} = \mathcal{J}_0 - \hat{\alpha}, \quad \mathbf{N} |n\rangle_\pm = n |n\rangle_\pm, \quad \hat{\alpha} |n\rangle_\pm = \alpha_\pm |n\rangle_\pm. \quad (6.21)$$

The ladder operators $\mathcal{J}_\pm = \mathcal{J}_1 \pm i\mathcal{J}_2$ act as

$$\mathcal{J}_+ |n\rangle_\pm = C_n^{\alpha_\pm} |n+1\rangle_\pm, \quad \mathcal{J}_- |n\rangle_\pm = C_{n-1}^{\alpha_\pm} |n-1\rangle_\pm, \quad |n\rangle_\pm \in \mathcal{F}_\pm, \quad (6.22)$$

$$C_n^{\alpha_\pm} = \sqrt{(2\alpha_\pm + n)(n+1)}, \quad (6.23)$$

cf. (3.9). The rising/lowering operators interchange the \mathcal{F}_+ and \mathcal{F}_- subspaces, $a^\pm : \mathcal{F}_+ \leftrightarrow \mathcal{F}_-$,

$$a^+ |n\rangle_+ = \sqrt{2(n+2\alpha_+)} |n\rangle_-, \quad a^+ |n\rangle_- = \sqrt{2(n+1)} |n+1\rangle_+, \quad (6.24)$$

$$a^- |n\rangle_+ = \sqrt{2n} |n-1\rangle_-, \quad a^- |n\rangle_- = \sqrt{2(n+2\alpha_+)} |n\rangle_+. \quad (6.25)$$

On the subspaces \mathcal{F}_\pm the $\mathfrak{so}(2, 1)$ spin is shifted by

$$\alpha_- - \alpha_+ = 1/2; \quad (6.26)$$

it is therefore legitimate to interpret the a^\pm as odd supercharges.

The deformed oscillator provides us with infinite-, and also finite, dimensional representations of $\mathfrak{osp}(1|2)$. Observe that, in (6.24) and (6.25), the coefficient $n + 2\alpha_+$ vanishes when n takes the value $n = -2\alpha_+ = -(1+\nu)/2$. This happens for odd-integer negative values of the deformation parameter, $\nu = -(2r+1)$, $r = 1, 2, \dots$, for which $n = r$. Then the Fock space becomes finite,

$$\mathcal{F} = \{|0\rangle, |1\rangle, \dots, |2r\rangle\} = \mathcal{F}_+ + \mathcal{F}_-, \quad (6.27)$$

$$\mathcal{F}_+ = \{|0\rangle_+ = |0\rangle, \dots, |r\rangle_+ = |2r\rangle\}, \quad \mathcal{F}_- = \{|0\rangle_- = |1\rangle, \dots, |r-1\rangle_- = |2r-1\rangle\}.$$

These spaces are characterized by the equations

$$a^{-(2r+1)} = a^{+(2r+1)} = 0, \quad a^- |0\rangle = 0, \quad a^+ |2r\rangle = 0, \quad \dim(\mathcal{F}) = |\nu| = 2r+1.$$

The spins carried by the subspaces \mathcal{F}_+ and \mathcal{F}_- are $\alpha_+ = -r/2$ and $\alpha_- = -r/2 + 1/2$, respectively.

For $\nu > -1$, the spectrum of \mathcal{J}_0 is positive definite and bounded from below, and the spin α_{\pm} takes only positive values. This produces the discrete series $D_{\alpha_{\pm}}^+$ of $\mathfrak{so}(2, 1)$. Every series is generated by the operators (6.17). The discrete series bounded from above, $D_{\alpha_{\pm}}^-$, of the Lorentz algebra are obtained by the external automorphism $(\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2) \rightarrow (-\mathcal{J}_0, \mathcal{J}_1, -\mathcal{J}_2)$, and projection to the corresponding subspaces \mathcal{F}_{\pm} .

For $\nu < -1$, $\nu \neq -(2r + 1)$, the RDHA provides us also with infinite dimensional half-bounded non-unitary irreducible representations $\tilde{D}_{\alpha_{\pm}}^{\pm}$ of $\mathfrak{so}(2, 1)$.

For the finite dimensional representation, the operators a^+ and a^- are conjugate, $(\Psi, a^-\Phi)^* = (\Phi, a^+\Psi)$, w.r.t. the scalar product,

$$(\Psi, \Phi) = \bar{\Psi}_n \Phi^n, \quad \bar{\Psi}_n = \Psi^{*k} \hat{\eta}_{kn}, \quad (6.28)$$

provided by the matrix $\hat{\eta} = \text{diag}(+1, -1, -1, +1, +1, \dots, (-1)^r, (-1)^r)$, where $\Phi^n = \langle n | \Phi \rangle$.

The two simplest examples of finite-dimensional representations of the reflection-deformed Heisenberg algebra (6.1), namely those of $\alpha_+ = -1/2$, $\alpha_- = 0$, and of $\alpha_+ = -1$, $\alpha_- = -1/2$, are summarized in the Appendix.

Note that for $\nu = -1$, the algebra (6.1) has a one-dimensional trivial representation, in which $R = 1$ and $a^{\pm} = 0$. This corresponds to the spin-0 representation of $\mathfrak{so}(2, 1)$.

The algebra (6.1) admits a nonlinear realization [36],

$$a^+ = \sqrt{1 + \frac{\nu}{\tilde{\mathcal{N}}}} \tilde{a}^+ \tilde{\Pi}_+ + \tilde{a}^+ \tilde{\Pi}_-, \quad a^- = \sqrt{1 + \frac{\nu}{\tilde{\mathcal{N}} + 1}} \tilde{a}^- \tilde{\Pi}_- + \tilde{a}^- \tilde{\Pi}_+, \quad (6.29)$$

in terms of the usual, non-deformed Heisenberg algebra of a bosonic oscillator,

$$[\tilde{a}^-, \tilde{a}^+] = 1, \quad [\tilde{\mathcal{N}}, \tilde{a}^{\pm}] = \pm \tilde{a}^{\pm}. \quad (6.30)$$

Here $\tilde{\mathcal{N}} = \tilde{a}^+ \tilde{a}^-$ is the number operator, and $\tilde{\Pi}_{\pm} = \frac{1}{2} (1 \pm \tilde{R})$, $\tilde{R} = (-1)^{\tilde{\mathcal{N}}}$, are projectors, $[\tilde{R}, \tilde{a}^{\pm}] = 0$. Consistently with (6.29) and (6.2), the number operators of the deformed, (6.1), and the non-deformed, (6.30), algebras coincide, $\mathcal{N} = \tilde{\mathcal{N}}$.

7 $N = 1$ supersymmetric generalization

Having written the various spin cases in a unified form, now we proceed to unify them into a supersymmetric theory. Consider in fact the following modification of Eq. (2.1),

$$V_{\mu}^{(\hat{\alpha})} \psi(x) = 0, \quad V_{\mu}^{(\hat{\alpha})} = \hat{\alpha} P_{\mu} - i \epsilon_{\mu\nu\lambda} P^{\nu} \mathcal{J}^{\lambda} + m \mathcal{J}_{\mu}. \quad (7.1)$$

Here \mathcal{J}_{μ} is the direct sum of irreducible representations of $\mathfrak{so}(2, 1)$, and $\hat{\alpha}$ is not more a c -number, but a (diagonal) *operator*, which takes the corresponding $\mathfrak{so}(2, 1)$ -spin value, α , on each $\mathfrak{so}(2, 1)$ -irreducible subspace,

$$\mathcal{J}_{\mu} \mathcal{J}^{\mu} = -\hat{\alpha}(\hat{\alpha} - 1), \quad (7.2)$$

cf. Eq. (2.16). The vector system (7.1) yields equations of the form (2.8)–(2.10) with β replaced by $\hat{\alpha}$. Note that the analog of Eq. (2.10) is automatically satisfied here, due to (7.2). Eq. (7.1) describes a multiplet of particles of the same mass, and the values of the Poincaré spins are given by the diagonal elements of $\hat{\alpha}$.

As found before (but in a less general framework) [27], choosing the direct sum of just two $\mathfrak{so}(2, 1)$ representations shifted by one half,

$$\hat{\alpha} = \text{diag}(\alpha, \alpha + 1/2), \quad (7.3)$$

provides us with an $N = 1$ supersymmetric system. To identify the supersymmetric structure of (7.1), (7.3), we realize the $\mathfrak{so}(2, 1)$ -generators quadratically in terms of the creation-annihilation operators of the RDHA, see (6.5). Consistently with (7.2), they satisfy the relation (6.13). They commute with the reflection operator R , identified as the \mathbb{Z}_2 -grading operator of the super-Poincaré structure we are looking for. The Fock space representation of the RDHA depends on the deformation parameter, and provides us with infinite- ($\nu \neq -(2r + 1)$), and $(2r + 1)$ -dimensional ($\nu = -(2r + 1)$, $r = 1, 2, \dots$) irreducible representations of the $\mathfrak{osp}(1|2)$ superalgebra. Each such representation is the direct sum of two irreducible $\mathfrak{so}(2, 1)$ representations with spin values shifted by $1/2$,

$$\alpha_+ = \frac{1}{4}(1 + \nu), \quad \alpha_- = \alpha_+ + \frac{1}{2}, \quad (7.4)$$

cf. (6.26), and consistently also with (7.3). These are just the values taken by the operator $\hat{\alpha}$ in (6.13), when restricted to the subspaces \mathcal{F}_\pm . The latter are invariant under the action of \mathcal{J}_μ , and are eigensubspaces of the reflection operator, see (6.14), (6.15). The wave function is expanded as

$$\psi(x) = \psi^+(x) + \psi^-(x), \quad \psi^\pm(x) = \sum_{n=0} \psi_n^\pm(x) |n\rangle_\pm, \quad |n\rangle_\pm \in \mathcal{F}_\pm, \quad (7.5)$$

and Eq. (7.1) is reduced to two independent equations,

$$V_\mu^{(\alpha_+)} \psi^+(x) = 0 \quad \text{and} \quad V_\mu^{(\alpha_-)} \psi^-(x) = 0, \quad (7.6)$$

whose solutions form a supermultiplet with spins α_+ and $\alpha_+ + 1/2 = \alpha_-$, respectively. The spinor supercharge of such an $N = 1$ -supersymmetric system is given by

$$Q_a = \frac{i}{\sqrt{2m(1 + \nu)}} \left((P^\mu \gamma_\mu)_a{}^b - m R \delta_a{}^b \right) \mathcal{L}_b, \quad a, b = 1, 2, \quad (7.7)$$

where the γ_μ 's are the Dirac matrices (6.9) in the Majorana representation. It is an operator-valued linear combination of the odd $\mathfrak{osp}(1|2)$ generators (6.6). To see that the (7.7) is indeed the supercharge, we calculate the commutator

$$[V_\mu^{(\hat{\alpha})}, Q_a] = \frac{i}{\sqrt{2m(1 + \nu)}} \left((D_\mu)_a{}^b \mathcal{D}_b \Pi_+ - \frac{1}{2} (\gamma_\mu)_a{}^b \mathcal{L}_b (P^2 + m^2) \right), \quad (7.8)$$

where

$$(D_\mu)_a{}^b = P_\mu \delta_a{}^b - m (\gamma_\mu)_a{}^b, \quad \mathcal{D}_b = (P_\mu (\gamma^\mu)_a{}^b - m \delta_a{}^b) \mathcal{L}_b, \quad (7.9)$$

and Π_+ is the projector on the even subspace \mathcal{F}_+ , see (6.19). The action of (7.8) on the wave functions that satisfy Eq. (7.1) produces zero if $\mathcal{D}_a \psi^+ = 0$. Such a spinor set of equations was studied in [32], where it was shown that its solutions describe a particle of mass m and spin α_+ .

Moreover, $V_\mu^{(\alpha_+)} \psi^+(x) = \frac{1}{4} \mathcal{L}^a (\gamma^\mu D_\mu)_a{}^b \mathcal{D}_b \psi^+(x)$, and any solution of the spinor set of equations in the even subspace is also a solution of the vector set.

We conclude, therefore, that (7.7) is indeed a supercharge. Together with the Poincaré generators, it yields the off-shell *nonlinear* superalgebra

$$[P_\mu, P_\nu] = 0, \quad [\mathcal{M}_\mu, P_\nu] = -i\epsilon_{\mu\nu\lambda}P^\lambda, \quad [\mathcal{M}_\mu, \mathcal{M}_\nu] = -i\epsilon_{\mu\nu\lambda}\mathcal{M}^\lambda, \quad (7.10)$$

$$[P_\mu, Q_a] = 0, \quad [\mathcal{M}_\mu, Q_a] = \frac{1}{2}(\gamma_\mu)_a{}^b Q_b, \quad (7.11)$$

$$\{Q_a, Q_b\} = -2i(P\gamma)_{ab} \quad (7.12)$$

$$+ \frac{2i}{m(1+\nu)} [(\mathcal{J}\gamma)_{ab}(P^2 + m^2) - 2(P\gamma)_{ab}(P\mathcal{J} - m\hat{\alpha})] . \quad (7.13)$$

The last term (7.13) vanishes on-shell, and (7.10)–(7.12) yield the usual $N = 1$ planar super-Poincaré algebra.

The deformed oscillator representation of the Lorentz algebra produces both the infinite half-bounded series, and the usual finite-dimensional series with integer or half-integer spin. Hence, the system of equations (7.1), (7.3) describes, universally, a supersymmetric system of massive particles of anyonic spin when $\nu \neq -(2r+1)$, $r = 1, 2, \dots$, and of usual integer/half-integer spin for $\nu = -3, -5, -7, \dots$. This is summarized in Table 1, where the notation (2.19) was used. Note that for $-3 < \nu < -1$, the supermultiplet includes anyons with spins of opposite signs, $-1/2 < \alpha_+ < 0$ and $\alpha_- = \alpha_+ + 1/2 > 0$, which are described in terms of infinite-dimensional half-bounded non-unitary and unitary representations, respectively.

Table 1: Poincaré supermultiplets

Deformation parameter	Poincaré spin supermultiplet	Lorentz rep.	Supermultiplet
$\nu \neq -(2r+1)$	(α_+, α_-)	$\mathcal{D}_{\alpha_+}^+ \oplus \mathcal{D}_{\alpha_-}^+$	anyons
$\nu = -(2r+1), r = 1, 2, \dots$	$(-r/2, -r/2 + 1/2)$	$\mathcal{D}_{-r/2}^+ \oplus \mathcal{D}_{-r/2+1/2}^+$	boson/fermion

Some examples of interest are listed in Table 2. Notice that the supermultiplet (a) was studied in a superfield approach in [30], see also Footnote 14. For the supermultiplet (c), the vector operator $V_\mu^{(\hat{\alpha})}$ vanishes identically on the spin-0 subspace. In this sector the Klein-Gordon equation should therefore be imposed, assigning to it $|0\rangle_-$ in the Fock space. Then the superpartner-fields are mapped into each other by the supercharge (7.7). Another possibility consists in taking $\nu < -3$, and then considering the limit $\nu \rightarrow -3$.

The supermultiplet (e) is studied in the next Section; the supermultiplets (f) and (g) can be considered in the same way. The analog of the DJT formulation [3] for the massive graviton was considered in [37, 38]; here we have the linearized form of the corresponding field equations.

7.1 Dirac/Jackiw-Deser-Templeton supermultiplet

To illustrate the general theory of the previous Section, we consider a supermultiplet composed of a *Dirac field*, paired with the *topological massive gauge vector field* of Deser, Jackiw and Templeton [3]. It is described by the vector system (7.1), where the spin operator $\hat{\alpha}$ has now eigenvalues -1 and $-1/2$, respectively. The deformed algebra (6.1) with deformation parameter $\nu = -5$ provides us with an $\mathfrak{osp}(1|2)$ representation, whose irreducible Lorentz components have

Table 2: Examples

Deformation parameter	Spin supermultiplet	Supermultiplet
$\nu = 0$	$(1/4, 3/4)$	(a) Quartions
$\nu = -2$	$(-1/4, 1/4)$	(b) Semion/quartion
$\nu = -3$	$(-1/2, 0)$	(c) Dirac/scalar
$\nu = -4$	$(-3/4, -1/4)$	(d) Semions
$\nu = -5$	$(-1, -1/2)$	(e) DJT/Dirac
$\nu = -7$	$(-3/2, -1)$	(f) Rarita-Schwinger/DJT
$\nu = -9$	$(-2, -3/2)$	(g) Massive graviton/gravitino

spin $j = -\alpha_+ = 1$ and $j = -\alpha_- = 1/2$, respectively. The superfield $\psi(x)$ is expanded in the $|\nu| = 5$ -dimensional basis (A.7),

$$\psi(x) = \begin{pmatrix} \psi_0^+(x) \\ \psi_0^-(x) \\ \psi_1^+(x) \\ \psi_1^-(x) \\ \psi_2^+(x) \end{pmatrix}. \quad (7.14)$$

The $\mathfrak{osp}(1|2)$ generators are 5×5 matrices given by Eqns. (A.14), (A.10) and (6.6). Using the projectors (A.9) and taking into account (A.13) and (A.15), we extract the components with spins 1 and $1/2$,

$$j = -\alpha_+ = 1 : \quad \psi^+(x) = \Pi_+ \psi(x) = \begin{pmatrix} \psi_0^+(x) \\ \psi_1^+(x) \\ \psi_2^+(x) \end{pmatrix}_+, \quad (7.15)$$

$$j = -\alpha_- = 1/2 : \quad \psi^-(x) = \Pi_- \psi(x) = \begin{pmatrix} \psi_0^-(x) \\ \psi_1^-(x) \end{pmatrix}_-. \quad (7.16)$$

The operator $V_\mu^{(\hat{\alpha})}$ in Eq. (7.1), projected to these components,

$$V_\mu^{(\hat{\alpha})} \Pi_\pm \psi(x) = V_\mu^{(\alpha_\pm)} \psi^\pm(x), \quad (7.17)$$

reduces to

$$V_\mu^{(\alpha_\pm)} = \alpha_\pm P_\mu + m J_\mu^{(\pm)} - i \epsilon_{\mu\nu\lambda} P^\nu J^{(\pm)\lambda}, \quad (7.18)$$

where,

$$J_0^{(-)} = -\frac{1}{2} \sigma_3, \quad J_1^{(-)} = \frac{i}{2} \sigma_1, \quad J_2^{(-)} = -\frac{i}{2} \sigma_2, \quad (7.19)$$

$$J_0^{(+)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad J_1^{(+)} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2^{(+)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (7.20)$$

$$J_\mu^{(-)} J^{(-)\mu} = -\frac{3}{4}, \quad J_\mu^{(+)} J^{(+)\mu} = -2. \quad (7.21)$$

The Pauli-Lubanski condition (2.9) implied by the equations (7.1) becomes, on one of the sectors, the Dirac equation¹⁸

$$(P\gamma' - m)_a{}^b(\psi^-(x))_b = 0, \quad \gamma'_\mu = -2J_\mu^{(-)}, \quad (7.22)$$

while, on the other sector, it becomes equivalent to the topologically massive gauge vector field equation¹⁹,

$$(P^\mu J_\mu^{(+)} + m)\psi^+(x) = 0.$$

Then we obtain (2.22), with

$$\left(-i\epsilon^\lambda{}_{\mu\nu}P^\mu + m\delta^\lambda{}_\nu\right)\psi^\nu = 0, \quad \psi^\nu = (U^\dagger\psi^+(x))^\nu, \quad -i\epsilon^\nu{}_{\mu\lambda} = (U^\dagger J_\mu^{(+)}U)^\nu{}_\lambda. \quad (7.23)$$

To see how the supercharge acts on this supermultiplet, it is convenient to consider the linear combinations

$$Q_\pm = Q_1 \mp iQ_2, \quad Q_\pm = \frac{i}{\sqrt{m(1+\nu)}} [\pm a^\mp P_\pm + a^\pm(mR \mp P_0)], \quad (7.24)$$

where $P_\pm = P_1 \pm iP_2$. Using the matrix representation (A.10) of the rising/lowering operators a^\pm , and the representation (A.9) of the reflection operator, the explicit action of the supercharges (7.24) on the spin-1 (ψ^+) and spin-1/2 (ψ^-) components of the supermultiplet is found,

$$Q_+\psi^-(x) = \frac{1}{\sqrt{m}} \begin{pmatrix} iP_+\psi_0^-(x) \\ \frac{i}{\sqrt{2}}P_+\psi_1^-(x) - \frac{1}{\sqrt{2}}(m+P_0)\psi_0^-(x) \\ -(m+P_0)\psi_1^-(x) \end{pmatrix}_+, \quad (7.25)$$

$$Q_-\psi^-(x) = \frac{1}{\sqrt{m}} \begin{pmatrix} i(P_0-m)\psi_0^-(x) \\ -\frac{1}{\sqrt{2}}P_-\psi_0^-(x) + \frac{i}{\sqrt{2}}(P_0-m)\psi_1^-(x) \\ -P_-\psi_1^-(x) \end{pmatrix}_+, \quad (7.26)$$

$$Q_+\psi^+(x) = \frac{1}{\sqrt{m}} \begin{pmatrix} \frac{1}{\sqrt{2}}P_+\psi_1^+(x) + i(m-P_0)\psi_0^+(x) \\ P_+\psi_2^+(x) + \frac{i}{\sqrt{2}}(m-P_0)\psi_1^+(x) \end{pmatrix}_-, \quad (7.27)$$

$$Q_-\psi^+(x) = \frac{1}{\sqrt{m}} \begin{pmatrix} -iP_-\psi_0^+(x) + \frac{1}{\sqrt{2}}(m+P_0)\psi_1^+(x) \\ -\frac{i}{\sqrt{2}}P_-\psi_1^+(x) + (m+P_0)\psi_2^+(x) \end{pmatrix}_-. \quad (7.28)$$

Here, it is explicitly shown how the components of the transformed spin-1 field depend on the untransformed spin-1/2 components (7.16) of the superfield (7.14).

Conversely, the components of the transformed spin-1/2 field are those of the spin 1 field (7.15).

¹⁸Note that the representation of Dirac matrices, γ'_μ in (7.22), is related to that in (6.9) by the unitary transformation (A.6).

¹⁹The standard (adjoint) representation of the spin-1 Lorentz algebra, $(J_\mu)^\nu{}_\lambda = -i\epsilon^\nu{}_{\mu\lambda}$, is obtained by the unitary transformation (A.17).

7.2 Jackiw-Nair/Majorana-Dirac supersymmetric system

Now, extending the Dirac-DJT correspondence established in the previous Section, we construct an $N = 1$ *supersymmetric unification* of the Jackiw-Nair and Majorana-Dirac anyonic fields. This can be done by combining the extended realization (5.1) and (5.2) [that corresponds to the sum of the $\mathfrak{so}(2,1)$ -representations] with the $N = 1$ supersymmetric scheme of Section 7. Taking the particular realization of the latter from Section 7.1 [i.e. taking the Dirac/DJT supermultiplet], we get, as a result, a particular, $N = 1$ supermultiplet of Majorana-Dirac and JN fields.

The generator J_μ in (5.1) is taken in an irreducible representation D_α^+ , while J'_μ is realized as a direct sum of two finite-dimensional $\mathfrak{so}(2,1)$ representations with Lorentz spins $j = 1$ and $j = 1/2$. Finite-dimensional representations can be obtained by means of the RDHA with deformation parameter $\nu = -5$. The parameter α' is promoted to a diagonal operator with eigenvalues -1 and $-1/2$. This immediately provides us with a supermultiplet of spins $\alpha - 1$ and $\alpha - 1/2$. On the corresponding subspaces with $\beta = \alpha - 1$ and $\beta = \alpha - 1/2$, our vector equations will take the form which corresponds to the Jackiw-Nair and Majorana-Dirac fields, respectively.

Explicitly, the $N = 1$ JN/MD supersymmetric system is described by the vector system of equations

$$V_\mu^{(\alpha+\hat{\alpha}')} \psi(x) = 0, \quad V_\mu^{(\alpha+\hat{\alpha}')} = (\alpha + \hat{\alpha}') P_\mu - i\epsilon_{\mu\nu\lambda} P^\nu \mathcal{J}^\lambda + m \mathcal{J}_\mu, \quad (7.29)$$

where

$$\mathcal{J}_\mu = J_\mu + \mathfrak{J}'_\mu, \quad J_\mu \in D_\alpha^+, \quad \mathfrak{J}'_\mu \in D^1 \oplus D^{1/2}, \quad \hat{\alpha}' = \text{diag}(-1, -1/2). \quad (7.30)$$

The five-dimensional irreducible representation of $\mathfrak{osp}(1|2)$ that corresponds to the chosen direct sum for \mathfrak{J}'_μ was described in the previous Section in terms of the RDHA with parameter $\nu = -5$. A solution of the system (7.29) is given by the JN field realized on $D_\alpha^+ \oplus \tilde{D}^1$, and by the MD field realized on $D_\alpha^+ \oplus \tilde{D}^{1/2}$. Here \tilde{D}^1 corresponds to the three-dimensional (vector) even subspace of five-dimensional Fock space and $\tilde{D}^{1/2}$ corresponds to its two-dimensional (spinor) odd subspace.

The spinor supercharge of such a supersymmetric system becomes

$$Q_a = \frac{1}{2\sqrt{2m}} \left((P^\mu \gamma_\mu)_a{}^b - m R \delta_a{}^b \right) \mathcal{L}_b, \quad a, b = 1, 2, \quad (7.31)$$

cf. (7.7). It is implied that (7.31) acts identically on the index n [not shown explicitly and corresponding to the infinite-dimensional representation D_α^+] of the wave function $\psi(x)$, while the matrix operator \mathcal{L}_b acts in the 5-dimensional irreducible representation of the $\mathfrak{osp}(1|2)$ superalgebra. The action of this supercharge is given by Eqns. (7.24), (7.25)–(7.28), where, in this case, it is implied that the components of the wave functions carry also [an not displayed] index n . Together with \mathcal{M}_μ and P_μ , this supercharge generates a nonlinear superalgebra of the form (7.10)–(7.13), with \mathcal{J}_μ in the last term (7.13) changed to \mathfrak{J}'_μ , and $\hat{\alpha}$ changed to $\hat{\alpha}'$.

It was shown in Section 2 that the vector sets of equations for JN and MD fields decouple into Majorana and DJT equations, and Majorana and Dirac equations, respectively. This means that the factor $(P\mathfrak{J}' - m\hat{\alpha}')$ disappears on-shell like (7.13). Therefore, on-shell, the standard $N = 1$ superalgebra given by Eqns. (7.10)–(7.12) is obtained.

8 $N = 2$ supersymmetry

In Section 7 we have shown that promoting the parameter α in (2.20) to a (diagonal) operator, $\hat{\alpha}$ in (7.3), and realizing the Lorentz generator \mathcal{J}_μ in terms of the RDHA creation-annihilation operators yields an $N = 1$ supersymmetric system of anyons, or of usual fields of arbitrary integer and half-integer spin.

In the previous Section we showed in turn that $N = 1$ supersymmetry can be obtained, alternatively, if, in the extended realization of Section 5, the second parameter α' is promoted to a diagonal operator, while α is left a fixed numerical parameter. The alternative construction allowed us to unify the Majorana-Dirac and JN fields in one anyon supermultiplet, and to get supermultiplets with partner anyon fields described by infinite-dimensional representations.

The construction of the previous section was asymmetric, and now we argue that the same prescription, applied to (5.2) *symmetrically* in the parameters α and α' , i.e., promoting them *both* to operators, provides us with an $N = 2$ -supersymmetric system of anyons, or of bosons and fermions. In such a way we unify, in particular, the Majorana-Dirac and Jackiw-Nair anyon systems into a single extended $N = 2$ supermultiplet.

Let us take

$$\mathcal{J}^\mu = \mathfrak{J}_\underline{1}^\mu + \mathfrak{J}_\underline{2}^\mu, \quad \mathfrak{J}_{\underline{A}\mu} \mathfrak{J}_\underline{A}^\mu = -\hat{\alpha}_\underline{A}(\hat{\alpha}_\underline{A} - 1), \quad \underline{A} = \underline{1}, \underline{2}, \quad (8.1)$$

[with no summation in \underline{A} implied], and postulate the vector system of equations

$$V_\mu^{(\hat{\beta})} \psi(x) = 0, \quad V_\mu^{(\hat{\beta})} = \hat{\beta} P_\mu + m \mathcal{J}_\mu - i \epsilon_{\mu\nu\lambda} P^\nu \mathcal{J}^\lambda, \quad (8.2)$$

$$\hat{\beta} = \hat{\alpha}_\underline{1} + \hat{\alpha}_\underline{2}. \quad (8.3)$$

Here we assume that each Lorentz generator $\mathfrak{J}_{\underline{A}\mu}$, $\underline{A} = \underline{1}, \underline{2}$, is realized quadratically in terms of the creation-annihilation operators of two independent RDHA's,

$$[a_\underline{A}^-, a_\underline{A}^+] = (1 + \nu_\underline{A} R_\underline{A}), \quad \{a_\underline{A}^\pm, R_\underline{A}\} = 0, \quad R_\underline{A}^2 = 1, \quad \underline{A} = \underline{1}, \underline{2}. \quad (8.4)$$

So, here

$$\hat{\alpha}_\underline{A} = \frac{1}{4}(2 + \nu_\underline{A} - R_\underline{A}), \quad (8.5)$$

$$\mathfrak{J}_{\underline{A}0} = \frac{1}{2} N_\underline{A} + \alpha_{\underline{A}+}, \quad \alpha_{\underline{A}+} = \frac{1}{4}(1 + \nu_\underline{A}). \quad (8.6)$$

In internal space we have two copies of $\mathfrak{osp}(1|2)$ superalgebra,

$$\mathfrak{osp}_\underline{1}(1|2) \oplus \mathfrak{osp}_\underline{2}(1|2), \quad \mathfrak{osp}_\underline{A}(1|2) = \{\mathcal{L}_{\underline{A}a}, \mathfrak{J}_\underline{A}^\mu\}, \quad \underline{A} = \underline{1}, \underline{2}, \quad (8.7)$$

which provide us with a representation of the Lorentz generators (8.1) with four irreducible components [two for each $\mathfrak{osp}_\underline{A}(1|2)$]. The \mathcal{J}_μ act on the tensor product of the Fock spaces,

$$\mathfrak{F} = \mathcal{F}_\underline{1} \otimes \mathcal{F}_\underline{2}, \quad \mathfrak{F} = \{|n_1, n_2\rangle = |n_1\rangle \otimes |n_2\rangle \mid |n_1\rangle \in \mathcal{F}_\underline{1}, |n_2\rangle \in \mathcal{F}_\underline{2}\}, \quad (8.8)$$

where $\mathcal{F}_\underline{A}$ is the Fock space of the corresponding RDHA. The wave function is expanded on \mathfrak{F} ,

$$\psi(x) = \sum_{n_1, n_2=0} \psi_{n_1, n_2}(x) |n_1, n_2\rangle. \quad (8.9)$$

The subdivision of the Fock spaces \mathcal{F}_A in even and odd subspaces,

$$\mathcal{F}_{A+} = \{|n_A\rangle_+ = |2n_A\rangle\}, \quad \mathcal{F}_{A-} = \{|n_A\rangle_- = |2n_A + 1\rangle\},$$

induces the decomposition of \mathfrak{F} ,

$$\mathfrak{F} = \mathcal{F}_{1+} \otimes \mathcal{F}_{2+} + \mathcal{F}_{1+} \otimes \mathcal{F}_{2-} + \mathcal{F}_{1-} \otimes \mathcal{F}_{2+} + \mathcal{F}_{1-} \otimes \mathcal{F}_{2-}. \quad (8.10)$$

The spin operator (8.3) takes, on the four corresponding subspaces in (8.10), the values

$$\hat{\beta} = \text{diag} \left(\chi, \chi + \frac{1}{2}, \chi + \frac{1}{2}, \chi + 1 \right), \quad \chi = \alpha_{1+} + \alpha_{2+} = \frac{1 + \nu_1}{4} + \frac{1 + \nu_2}{4}. \quad (8.11)$$

Observe that the spin eigenvalue $\chi + \frac{1}{2}$ is doubly degenerated. The wave function (8.9) has therefore four components,

$$\psi(x) = \psi^{++}(x) + \psi^{+-}(x) + \psi^{-+}(x) + \psi^{--}(x), \quad (8.12)$$

$$\psi^{\pm\pm}(x) = \sum_{n_1, n_2=0} \psi_{n_1, n_2}^{\pm\pm}(x) |n_1, n_2\rangle_{\pm\pm}, \quad (8.13)$$

$$\psi^{\pm\mp}(x) = \sum_{n_1, n_2=0} \psi_{n_1, n_2}^{\pm\mp}(x) |n_1, n_2\rangle_{\pm\mp}, \quad (8.14)$$

$$|n_1, n_2\rangle_{\pm\pm} \in \mathcal{F}_{1\pm} \otimes \mathcal{F}_{2\pm}, \quad |n_1, n_2\rangle_{\pm\mp} \in \mathcal{F}_{1\mp} \otimes \mathcal{F}_{2\mp}. \quad (8.15)$$

The wave function (8.12) solves (8.2) if each of its component solves, independently,

$$\begin{aligned} V_\mu^{(\chi)} \psi^{++}(x) &= 0, \\ V_\mu^{(\chi+1/2)} \psi^{+-}(x) &= 0, \\ V_\mu^{(\chi+1/2)} \psi^{-+}(x) &= 0, \\ V_\mu^{(\chi+1)} \psi^{--}(x) &= 0. \end{aligned} \quad (8.16)$$

Each component carries therefore an irreducible representation of the Poincaré group with mass m and spin indicated by the upper index of the vector operator.

Fields which differ by one half in their spins can be connected by supercharges analogous to (7.7). They are given by

$$Q_{1a} = \frac{i}{\sqrt{2m(1 + \nu_1)}} \left((P^\mu \gamma_\mu)_a{}^b - m R_{1\perp} \delta_a{}^b \right) \mathcal{L}_{1b}, \quad (8.17)$$

$$Q_{2a} = \frac{i}{\sqrt{2m(1 + \nu_2)}} R_{1\perp} \left((P^\mu \gamma_\mu)_a{}^b - m R_{2\perp} \delta_a{}^b \right) \mathcal{L}_{2b}. \quad (8.18)$$

The additional factor $R_{1\perp}$ is included into the second supercharge in order to make (8.17) and (8.18) anti-commute, see (8.21) below. The action of the supercharges on the supermultiplet components is illustrated on Fig. 1.

To see that (8.17) and (8.18) are physical operators in that they preserve the physical subspace, we pass to the rest frame ²⁰, $P^\mu = (m, 0, 0)$, in which the vector system is equivalent to $(\mathcal{J}_0 - \hat{\beta})\psi(x) = 0$, $\mathcal{J}_-\psi(x) = 0$, cf. (2.14). Observe that

$$\mathcal{J}_-|0, 0\rangle_{\pm\pm} = 0, \quad \mathcal{J}_-|0, 0\rangle_{\pm\mp} = 0,$$

²⁰It is implied that at least one of the parameters ν_A , $A = 1, 2$, corresponds to an infinite-dimensional representation of RDHA ($\nu > -1$).

$$\begin{array}{ccc}
[\psi^{+-}(x); s = \chi + 1/2] & \xleftarrow{Q_{\underline{1}}} & [\psi^{--}(x); s = \chi + 1] \\
\uparrow & & \uparrow \\
Q_{\underline{2}} & & Q_{\underline{2}} \\
\downarrow & & \downarrow \\
[\psi^{++}(x); s = \chi] & \xleftarrow{Q_{\underline{1}}} & [\psi^{-+}(x); s = \chi + 1/2]
\end{array}$$

Figure 1: *The action of supercharges on the supermultiplet components. Each supercharge, $Q_{\underline{1}}$ and $Q_{\underline{2}}$, changes the spin by $1/2$; the first (second) supercharge acts nontrivially in the first (second) index of the Fock space decomposition (8.9).*

and that $|0,0\rangle_{++} = |0,0\rangle$, $|0,0\rangle_{+-} = |0,1\rangle$, $|0,0\rangle_{-+} = |1,0\rangle$, $|0,0\rangle_{--} = |1,1\rangle$. The same set of kets is annihilated also by the operator $\mathcal{J}_0 - \hat{\beta}$. In the rest frame, the general solution of (8.2) is, therefore, an arbitrary linear combination of the four lower ket states,

$$\psi(x) \propto \psi_{0,0}^{++}|0,0\rangle_{++} + \psi_{0,0}^{+-}|0,0\rangle_{+-} + \psi_{0,0}^{-+}|0,0\rangle_{-+} + \psi_{0,0}^{--}|0,0\rangle_{--},$$

multiplied by a time-dependent phase [not displayed here]. The coefficients are arbitrary constants. In the rest frame, the operators (8.17) and (8.18) are proportional to $a_{\underline{A}}^+ \Pi_{\underline{A}+}$ and $a_{\underline{A}}^- \Pi_{\underline{A}-}$ with values of index $\underline{A} = \underline{1}$ and $\underline{2}$, respectively.

$$\begin{array}{ccc}
|0,1\rangle & \xrightarrow{a_{\underline{1}}^+} \longrightarrow & |1,1\rangle \\
& \xleftarrow{a_{\underline{1}}^-} \longleftarrow & \\
a_{\underline{2}}^+ \uparrow \downarrow a_{\underline{2}}^- & & a_{\underline{2}}^+ \uparrow \downarrow a_{\underline{2}}^- \\
|0,0\rangle & \xrightarrow{a_{\underline{1}}^+} \longrightarrow & |1,0\rangle \\
& \xleftarrow{a_{\underline{1}}^-} \longleftarrow &
\end{array} \tag{8.19}$$

Figure 2: *In the rest frame $P^\mu = (m, 0, 0)$, the action of the supercharges, shown in Fig. 1, is reduced to that of creation-annihilation operators.*

Their action on solutions is reduced to that of the creation and annihilation operators, as shown on Figure 2. Therefore, the space of solutions is invariant under the action of (8.17) and (8.18). They are odd operators with respect to $\Gamma = R_{\underline{1}}R_{\underline{2}}$, identified as the \mathbb{Z}_2 -grading operator of the $N = 2$ Poincaré superalgebra. The even part of this superalgebra is given by Eq. (7.10). The part that involves the supercharges is

$$[P_\mu, Q_{\underline{A}a}] = 0, \quad [\mathcal{J}_\mu, Q_{\underline{A}a}] = \frac{1}{2}(\gamma_\mu)_a{}^b Q_{\underline{A}b}, \tag{8.20}$$

$$\{Q_{\underline{A}a}, Q_{\underline{B}b}\} = -2i\delta_{\underline{AB}}(P\gamma)_{ab}, \tag{8.21}$$

where the anti-commutator of the supercharges is shown in on-shell form.

When both parameters $\nu_{\underline{A}}$, $\underline{A} = 1, 2$, correspond to finite-dimensional representations of RDHA, the equations (8.2), (8.3) provide us with a usual, boson/fermion $N = 2$ super-multiplet (8.11), with states of both signs of the energy.

Jackiw-Nair/Majorana-Dirac $N = 2$ supermultiplet

The Lorentz generator (8.1), $\mathcal{J}^\mu = \mathfrak{J}_1^\mu + \mathfrak{J}_2^\mu$, is constructed in terms of two RDHA algebras, one for \mathfrak{J}_1^μ and another for \mathfrak{J}_2^μ . We have the freedom to choose the deformation parameters ν_1 and ν_2 independently. As a result, the system of equations (8.2) can describe a variety of $N = 2$ supermultiplets (8.11), based on different representations of the $\mathfrak{so}(2, 1)$ algebra, by mixing “anyonic” and/or usual integer/half-integer spin series.

As an example, we consider here the Jackiw-Nair/Majorana-Dirac $N = 2$ supermultiplet. It is realized by choosing $\nu_1 > -1$ and $\nu_2 = -5$. This implies,

$$\alpha_{1+} > 0, \quad -\alpha_{2+} = j_2 = 1 \quad \Rightarrow \quad \chi = \alpha_{1+} - 1. \quad (8.22)$$

Hence we get the spectrum described in Table 3.

Table 3: $N = 2$ JN/MD supermultiplet

Field component	Fine spin structure	Spin s	Internal space	Field type
$\psi^{++}(x)$	$\alpha_{1+} - 1$	$\alpha_{1+} - 1$	$\mathcal{F}_{1+}^{(\infty)} \otimes \mathcal{F}_{2+}^{(3)}$	JN
$\psi^{+-}(x)$	$\alpha_{1+} - 1/2$	$\alpha_{1+} - 1/2$	$\mathcal{F}_{1+}^{(\infty)} \otimes \mathcal{F}_{2-}^{(2)}$	MD
$\psi^{-+}(x)$	$\alpha_{1-} - 1$	$\alpha_{1+} - 1/2$	$\mathcal{F}_{1-}^{(\infty)} \otimes \mathcal{F}_{2+}^{(3)}$	JN
$\psi^{--}(x)$	$\alpha_{1-} - 1/2$	α_{1+}	$\mathcal{F}_{1-}^{(\infty)} \otimes \mathcal{F}_{2-}^{(2)}$	MD

It is interesting to note that, from the viewpoint of the Poincaré spin value s [see the third column in the Table], the field component ψ^{--} looks like the field we got in Section 2 using the irreducible representation D_α^+ with $\alpha = \alpha_{1+} > 0$. The components ψ^{-+} and ψ^{+-} look like anyonic fields produced by the Majorana-Dirac system. Finally, the component ψ^{++} looks like that described by the Jackiw-Nair equations. Having in mind, however, that the equations for each field component have a structure that corresponds to the scheme (5.1), (5.2) based on the direct sum of two irreducible representations of $\mathfrak{so}(2, 1)$ (see the second and fourth columns), each component is interpreted as a field of the type indicated in the fifth column.

The wave functions have two Lorentz indices, associated to $\mathcal{F}_1^{(\infty)} = \mathcal{F}_{1-}^{(\infty)} + \mathcal{F}_{1+}^{(\infty)}$ (infinite

dimensional) and $\mathcal{F}_2^{(5)} = \mathcal{F}_{2-}^{(2)} + \mathcal{F}_{2+}^{(3)}$ (5-dimensional) representations of RDHA. Explicitly,

$$\psi^{++}(x) = \sum_{n=0}^{\infty} \sum_{\lambda'=1,2,3} (\psi_n^{++}(x))^{\lambda'} |n, \lambda'\rangle_{++}, \quad |n, \lambda'\rangle_{++} \in \mathcal{F}_{1+}^{(\infty)} \otimes \mathcal{F}_{2+}^{(3)}, \quad (8.23)$$

$$\psi^{+-}(x) = \sum_{n=0}^{\infty} \sum_{a=1,2} (\psi_n^{+-}(x))^a |n, a\rangle_{+-}, \quad |n, a\rangle_{+-} \in \mathcal{F}_{1+}^{(\infty)} \otimes \mathcal{F}_{2-}^{(2)}, \quad (8.24)$$

$$\psi^{-+}(x) = \sum_{n=0}^{\infty} \sum_{\lambda'=1,2,3} (\psi_n^{-+}(x))^{\lambda'} |n, \lambda'\rangle_{-+}, \quad |n, \lambda'\rangle_{-+} \in \mathcal{F}_{1-}^{(\infty)} \otimes \mathcal{F}_{2+}^{(3)}, \quad (8.25)$$

$$\psi^{--}(x) = \sum_{n=0}^{\infty} \sum_{a=1,2} (\psi_n^{--}(x))^a |n, a\rangle_{--}, \quad |n, a\rangle_{--} \in \mathcal{F}_{1-}^{(\infty)} \otimes \mathcal{F}_{2-}^{(2)}. \quad (8.26)$$

Here, $a = 1, 2$, is a spinor index in the basis (A.13), in which the $\mathfrak{so}(2, 1)$ generators are given by Eq. (A.5). The vector index λ' corresponds to the basis (A.15), and transforms under the $\mathfrak{so}(2, 1)$ representation (A.14). Index n transforms under the infinite dimensional $\mathfrak{so}(2, 1)$ representation ($D_{\alpha_{1\pm}}^+$), and endows a wave function with the anyonic part of the spin. Supercharges (8.17), (8.18) transform these fields as shown on Fig. 1.

9 Nonrelativistic limit

In this Section we study the non-relativistic limit of our vector system of equations (2.1). This will provide us with a universal non-relativistic description of either fermion/boson fields of arbitrary half-integer/integer spin, or of non-unitary anyons interpolating between them, or of anyons based on unitary representations. As shown below, the system of linear differential equations we get implies, for each component, the Schrödinger equation as integrability condition – just like the Klein-Gordon equation is implied in the relativistic case. Also, the equations guarantee that the system carries an irreducible representation of the corresponding Galilei symmetry.

We analyze two different types of non-relativistic limits [22].

- The first one is the usual limit, in which the velocity of light, c , tends to infinity, while the remaining parameters are kept constant. This generalizes the *Lévy-Leblond equations* valid for spin 1/2 [21] to arbitrary spin.

- In the second, “*exotic*” type of limit introduced by Jackiw and Nair [23] (see also [24, 22]) the spin, s , also goes to infinity in such a way that a ratio

$$c \rightarrow \infty, \quad s \rightarrow \infty, \quad s/c^2 = \kappa \quad (9.1)$$

remains constant. Its “*raison d’être*” is that it yields *exotic Galilei symmetry* [39], which admits, besides the mass, m , also a second central charge, κ , associated with the non-commutativity of Galilean boosts [39, 40].

Both non-relativistic limits are conveniently derived from (2.1), after reinstating the velocity of light, c , and substituting $m \rightarrow mc$ and $\psi \rightarrow e^{-imc^2 t} \psi$, which gives

$$P^0 = \frac{1}{c} \left(i \frac{\partial}{\partial t} + mc^2 \right). \quad (9.2)$$

Then Eqns. (4.4) and (4.5) take the equivalent form,

$$\frac{1}{c}\sqrt{n+2\alpha}\left(i\frac{\partial}{\partial t}\right)\psi_n + \sqrt{n+1}P_+\psi_{n+1} = 0, \quad (9.3)$$

$$\sqrt{n+2\alpha}P_-\psi_n + \sqrt{n+1}\left(2mc + \frac{1}{c}\left(i\frac{\partial}{\partial t}\right)\right)\psi_{n+1} = 0, \quad (9.4)$$

while Eq. (4.6) reduces to

$$\left(-\alpha\frac{1}{c}i\frac{\partial}{\partial t} + mcn\right)\psi_n + \frac{1}{2}\left(\sqrt{(2\alpha+n-1)n}P_-\psi_{n-1} - \sqrt{(2\alpha+n)(n+1)}P_+\psi_{n+1}\right) = 0. \quad (9.5)$$

The Klein-Gordon equation, a consequence of Eqns. (9.3)–(9.5), takes here the equivalent form

$$\left(\left(1 + \frac{i}{2mc^2}\partial_t\right)i\partial_t - \frac{1}{2m}\vec{P}^2\right)\psi = 0. \quad (9.6)$$

We put

$$K_i = -\frac{1}{c}\epsilon_{ij}\mathcal{M}_j = -tP_i + mx_i - \epsilon_{ij}\mathcal{J}_j + \frac{1}{c^2}x_i \cdot i\partial_t, \quad (9.7)$$

$$J = \mathcal{M}_0 = \epsilon_{ij}x_iP_j + \mathcal{J}_0, \quad (9.8)$$

and get, for the commutator of the boosts,

$$[K_i, K_j] = -i\frac{1}{c^2}\epsilon_{ij}J, \quad (9.9)$$

where $\epsilon_{ij} = -\epsilon_{ji}$, $\epsilon_{12} = 1$.

Let us insist that all these formulas are still relativistic; we simply wrote them in a form where the role of the speed of light is highlighted.

9.1 Usual non-relativistic limit

From Eq. (9.4) we infer that, as $c \rightarrow \infty$, every subsequent component is $O(\frac{1}{c})$ -times the previous one. The case of a boson/fermion of nonzero spin $j = n/2 > 0$ is described by a $(2j+1)$ -component field. Considering the non-relativistic limit, one could try to preserve all terms up to order $1/c^n$, $n > 1$. Though such an approximation would be rather natural for boson/fermion fields of corresponding spin, it will not possess a property of universality. Indeed, particularly, as it follows from (9.6), the Hamiltonian operator in the Schrödinger equation then will contain corrections with terms of the form $(\vec{P}^2/mc^2)^k$, for which the highest value of k will depend on the value of spin. Also, according to (9.9), the boost generators will commute for $j = 1/2$, but will be non-commuting for $j > 1/2$. The question is then: what kind of symmetry will underlie the resulting theory?

Investigating the general question goes beyond our scope here; we consider, instead, a certain non-relativistic limit which is characterized by the following properties:

- It has a universal structure; in particular, the dynamics is described by the Schrödinger equation with a Hamiltonian operator having the usual non-relativistic form;
- The corresponding Lie-algebraic symmetry is produced by Inönü-Wigner contraction from Poincaré symmetry;

- For $j = 1/2$, it reproduces the Lévy-Leblond theory for a non-relativistic spin one-half field.

Let us denote the multicomponent field by Ψ , and consider the (invertible) similarity transformation $\Psi \rightarrow \Phi$,

$$\Phi = \mathbf{M}\Psi, \quad \mathbf{M} = \text{diag}(1, c^1, c^2, \dots). \quad (9.10)$$

The dimension of the field-column

$$\Phi = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \end{pmatrix}, \quad (9.11)$$

and of diagonal matrix \mathbf{M} depends on the chosen representation of the Lorentz algebra; in the anyon case, it is infinite. The components of the transformed field Φ , unlike those of Ψ , are of order $c^0 = 1$ cf. Eq. (9.4).

The transformation (9.10) induces a similarity transformation of operators : to any operator \mathfrak{D} acting on a state Ψ , there corresponds an operator

$$\check{\mathfrak{D}} = \mathbf{M}\mathfrak{D}\mathbf{M}^{-1} \quad (9.12)$$

that acts on Φ . For the translational invariant (spin) part of the Lorentz generators, $\mathfrak{D} = \mathcal{J}_0, \mathcal{J}_+, \mathcal{J}_-$, we get then

$$\check{\mathcal{J}}_0 = \mathcal{J}_0, \quad \check{\mathcal{J}}_+ = c\mathcal{J}_+, \quad \check{\mathcal{J}}_- = \frac{1}{c}\mathcal{J}_-. \quad (9.13)$$

The components of the vector operator V_μ of our basic equations are transformed into

$$\check{V}_0 = -\frac{1}{c} \left(\alpha i\partial_t + \frac{1}{2}P_+\mathcal{J}_- \right) + c \left(m(\mathcal{J}_0 - \alpha) + \frac{1}{2}P_-\mathcal{J}_+ \right), \quad (9.14)$$

$$\check{V}_+ = -(\mathcal{J}_0 - \alpha)P_+ - \mathcal{J}_+i\partial_t, \quad (9.15)$$

$$\check{V}_- = (\mathcal{J}_0 + \alpha)P_- + 2m\mathcal{J}_- + \frac{1}{c^2}i\partial_t. \quad (9.16)$$

So far we merely deformed the theory, which is still relativistic. To obtain the non-relativistic limit of the equations presented in terms of the rescaled field Φ , it is necessary to ‘renormalize’ \check{V}_0 and consider

$$\mathfrak{V}_0 = \lim_{c \rightarrow \infty} \frac{1}{c} \check{V}_0 = m(\mathcal{J}_0 - \alpha) + \frac{1}{2}\mathcal{J}_+P_-, \quad (9.17)$$

$$\mathfrak{V}_+ = -\lim_{c \rightarrow \infty} \check{V}_+ = (\mathcal{J}_0 - \alpha)P_+ + \mathcal{J}_+i\partial_t, \quad (9.18)$$

$$\mathfrak{V}_- = \lim_{c \rightarrow \infty} \check{V}_- = (\mathcal{J}_0 + \alpha)P_- + 2m\mathcal{J}_-. \quad (9.19)$$

Then we infer our new vector equations

$$\mathfrak{V}_0\Phi = 0, \quad \mathfrak{V}_+\Phi = 0, \quad \mathfrak{V}_-\Phi = 0. \quad (9.20)$$

The dynamics of the field Φ is given by the Schrödinger equation, $i\partial_t\Phi = \frac{1}{2m}\vec{P}^2\Phi$. The latter appears in fact as consistency (integrability) condition for the system (9.20).

In component form the last two equations read

$$\sqrt{n+2\alpha} \left(i \frac{\partial}{\partial t} \right) \phi_n + \sqrt{n+1} P_+ \phi_{n+1} = 0, \quad (9.21)$$

$$\sqrt{n+2\alpha} P_- \phi_n + 2m\sqrt{n+1} \phi_{n+1} = 0, \quad (9.22)$$

where $n = 0, 1, \dots, \infty$ for anyons ($\alpha > 0$, or $0 > \alpha \neq -j$), and $n = 0, 1, \dots, 2j$ for bosons/fermions ($\alpha = -j$). The component form of the first equation from (9.20) is reduced here to Eq. (9.22). Expressing ϕ_{n+1} from Eq. (9.21),

$$\phi_{n+1} = -\frac{1}{2m} \sqrt{\frac{n+2\alpha}{n+1}} P_- \phi_n. \quad (9.23)$$

Note that the tower of states is automatically finite as it should if 2α is a negative integer. Inserting (9.23) into (9.22) shows explicitly that each component satisfies, separately, the Schrödinger equation,

$$\left(i \frac{\partial}{\partial t} - \frac{\vec{P}^2}{2m} \right) \phi_n = 0. \quad (9.24)$$

Iterating (9.23) allows us to express all components ϕ_n in terms of the lowest one,

$$\phi_n = (nB(2\alpha, n))^{-1/2} \left(\frac{-P_-}{2m} \right)^n \phi_0, \quad (9.25)$$

cf. (4.8). By Eq. (9.24), ϕ_0 is a (superposition of) plane waves,

$$\phi_0 = \exp \left\{ -it \frac{\vec{p}^2}{2m} + i\vec{x} \cdot \vec{p} \right\}. \quad (9.26)$$

Note that for integer/half-integer spin $j = -\alpha = 1/2, 1, 3/2, \dots$, Eqns. (9.23)-(9.25) yield $\phi_n = 0$ for $n > 2j$, consistently with our expectation that spin- j particle should be described by a $(2j+1)$ -component field, cf. Section 3.

As in the relativistic case, the infinite-component anyon fields of spin $s = \alpha$ given in Eq. (3.14), interpolate between the finite-component non-relativistic fields of spins $j - 1/2$ and j , respectively. Here, however, each higher component ϕ_n is suppressed by the additional hidden factor $\frac{1}{c^n}$ w.r.t. the leading component ϕ_0 . When α tends to either of the boundary values $-j + 1/2$ or $-j$, the components ϕ_n with $n > 2j - 1$ or $n > 2j + 1$ are suppressed, in addition, by the numerical factor $(j - \frac{1}{2} + \alpha)^{1/2}$ or $(j + \alpha)^{1/2}$, see Section 3. In the case of anyons based on the unitary representations D_α^+ with $\alpha > 0$, no such additional suppression arises.

Now we show that the system (9.21), (9.22) has a 1-parameter centrally extended Galilei symmetry, obtained by Inönü-Wigner contraction from the $(2+1)$ D Poincaré algebra,

$$[P_\mu, P_\nu] = 0, \quad [\mathcal{M}_\mu, P_\nu] = -i\epsilon_{\mu\nu\lambda} P^\lambda, \quad [\mathcal{M}_\mu, \mathcal{M}_\nu] = -i\epsilon_{\mu\nu\lambda} \mathcal{M}^\lambda. \quad (9.27)$$

The Galilean boost generators are defined by

$$\mathcal{K}_i = -\lim_{c \rightarrow \infty} \epsilon_{ij} \tilde{\mathcal{M}}_j / c. \quad (9.28)$$

With taking into account Eqns. (9.7) and (9.13), we get

$$\mathcal{K}_\pm = \mathcal{K}_1 \pm i\mathcal{K}_2 = -tP_\pm + mx_\pm + \Delta_\pm, \quad \Delta_- = 0, \quad \Delta_+ = i\mathcal{J}_+. \quad (9.29)$$

Note here the presence of the spin-dependent part Δ_+ .

The generator of rotations has, by (9.7), (9.13), the same form as in relativistic case,

$$\mathbf{J} = \epsilon_{ij} x_i P_j + \mathcal{J}_0. \quad (9.30)$$

Defining also

$$\mathcal{H} = cP^0 - mc^2, \quad (9.31)$$

we find that P_i , \mathcal{K}_i , \mathbf{J} and $\mathcal{H} = i\frac{\partial}{\partial t}$ are symmetry operators of the system (9.21), (9.22), and generate the extended Galilei (“Bargmann”) algebra with a unique central charge m ,

$$[\mathcal{K}_i, P_j] = im\delta_{ij}, \quad [P_i, P_j] = 0, \quad [\mathcal{H}, P_i] = 0, \quad [\mathcal{K}_i, \mathcal{K}_j] = 0, \quad (9.32)$$

$$[\mathcal{K}_i, \mathcal{H}] = iP_i, \quad [\mathbf{J}, P_i] = i\epsilon_{ij} P_j, \quad [\mathbf{J}, \mathcal{K}_i] = i\epsilon_{ij} \mathcal{K}_j. \quad (9.33)$$

Note that by (9.9) boosts commute.

The Casimir operators of the algebra (9.32), (9.33) are

$$\mathcal{C}_1 = P_i^2 - 2m\mathcal{H}, \quad \mathcal{C}_2 = m\mathbf{J} - \epsilon_{ij} \mathcal{K}_i P_j. \quad (9.34)$$

On-shell, i.e. on the surface of Eqns. (9.21), (9.22), they take the values

$$\mathcal{C}_1 = 0, \quad \mathcal{C}_2 = \alpha m. \quad (9.35)$$

The non-relativistic equations (9.21), (9.22) describe therefore a massive non-relativistic particle of spin α .

The particular cases of spin 1/2 and spin 1 will be discussed in Section 10.3.

9.2 Exotic limit

For the infinite-dimensional unitary representations D_α^+ , $\alpha > 0$, we define the operators,

$$\mathbf{N} = \mathcal{J}_0 - \alpha, \quad b^\pm = \frac{\mathcal{J}_\pm}{\sqrt{2\alpha}}, \quad (9.36)$$

and put $\alpha \rightarrow \infty$. In this limit, the operators (9.36) generate the harmonic oscillator algebra,

$$[\mathbf{N}, b^\pm] = b^\pm, \quad [b^-, b^+] = 1, \quad (9.37)$$

and the $\mathfrak{so}(2, 1)$ representation (3.10) transforms into a Fock space representation,

$$\mathbf{N}|n\rangle = n|n\rangle, \quad b^-|n\rangle = \sqrt{n+1}|n+1\rangle, \quad b^+|n\rangle = \sqrt{n}|n-1\rangle, \quad n = 0, 1, 2, \dots \quad (9.38)$$

The exotic non-relativistic limit $c \rightarrow \infty$, $\alpha \rightarrow \infty$, $\alpha/c^2 = \kappa = \text{const}$, cf. (9.1), applied to the equations (4.2)–(4.3) with the substitution (9.2), produces

$$\Lambda_0 \psi = 0, \quad \Lambda_+ \psi = 0, \quad \Lambda_- \psi = 0, \quad (9.39)$$

$$\Lambda_0 = i\frac{\partial}{\partial t} - \frac{v_- P_+}{2} + \frac{v_+}{2} \Lambda_-, \quad \Lambda_+ = \kappa v_+ \left(i\frac{\partial}{\partial t} - \frac{1}{2} v_- P_+ \right), \quad \Lambda_- = P_- - mv_- . \quad (9.40)$$

Here, Λ_0 , Λ_- and Λ_+ correspond to the limit (9.1) of the operators $-cV_0/\alpha$, $V_-/2\alpha$ and V_+ , respectively, and we have introduced the notation

$$v_\pm = v_1 \pm iv_2 = -\sqrt{\frac{2}{\kappa}} b^\pm. \quad (9.41)$$

Taking the combination $\Lambda_0 - v_+ \Lambda_-$, we get a minimal set of equations,

$$\left(i \frac{\partial}{\partial t} - H\right) \psi = 0, \quad \Lambda_- \psi = 0, \quad (9.42)$$

where

$$H = \vec{P} \vec{v} - \frac{m}{2} v_+ v_- \quad (9.43)$$

is identified as the [linear-in- P] *Hamiltonian operator*.

Taking into account the second equation from (9.42), the first one becomes the Schrödinger equation,

$$\left(i \frac{\partial}{\partial t} - \frac{1}{2m} \vec{P}^2\right) \psi = 0. \quad (9.44)$$

On the other hand, decomposing as $\psi = \sum_n \psi_n |n\rangle$, the equations $\Lambda_{\pm} \psi = 0$ read

$$\sqrt{2\kappa} \left(i \frac{\partial}{\partial t}\right) \psi_n + \sqrt{n+1} P_+ \psi_{n+1} = 0, \quad (9.45)$$

$$\sqrt{2\kappa} P_- \psi_n + 2m\sqrt{n+1} \psi_{n+1} = 0. \quad (9.46)$$

These equations can also be obtained, alternatively, by applying the exotic limit (9.1) to Eqns. (9.3), (9.4). Using the second equation from (9.46), the higher components can be expressed in terms of the lowest one, namely as

$$\psi_n = \frac{1}{\sqrt{n!}} \left(-\sqrt{\frac{\kappa}{2}} \frac{P_-}{m}\right)^n \psi_0,$$

while, according to the first equation, the dynamics of each component is governed by the Schrödinger equation, (9.44).

The exotic limit with the additional ‘renormalization’

$$\mathcal{M}_0 \rightarrow \mathcal{M}_0 - \alpha = \mathcal{M}_0 - \kappa c^2, \quad (9.47)$$

cf. (9.31), provides us with rotation and Galilean boosts generators,

$$\mathbf{J} = \epsilon_{ij} x_i P_j + \frac{1}{2} \kappa v_+ v_-, \quad \mathcal{K}_i = m x_i - t P_i + \kappa \epsilon_{ij} v_j. \quad (9.48)$$

Together with $\mathcal{H} = i \frac{\partial}{\partial t}$ and momentum P_i , they generate the exotic Galilei symmetry,

$$[\mathcal{K}_i, P_j] = i m \delta_{ij}, \quad [P_i, P_j] = 0, \quad [\mathcal{H}, P_i] = 0, \quad [\mathcal{K}_i, \mathcal{K}_j] = -i \kappa \epsilon_{ij}, \quad (9.49)$$

$$[\mathcal{K}_i, \mathcal{H}] = i P_i, \quad [\mathbf{J}, P_i] = i \epsilon_{ij} P_j, \quad [\mathbf{J}, \mathcal{K}_i] = i \epsilon_{ij} \mathcal{K}_j. \quad (9.50)$$

In particular, the parameter κ , which measures the non-commutativity of boosts, becomes the second central charge, highlighting the exotic Galilean symmetry [39, 40]. The Casimir operators are

$$\mathcal{C}_1 = P_i^2 - 2m\mathcal{H}, \quad \mathcal{C}_2 = m\mathbf{J} - \epsilon_{ij} \mathcal{K}_i P_j + \kappa \mathcal{H}. \quad (9.51)$$

On-shell, they take the values $\mathcal{C}_1 = \mathcal{C}_2 = 0$. The system of equations (9.42) describes an exotic (2+1)D non-relativistic particle with mass m and the exotic parameter κ . The system may be reinterpreted as a free massive particle on the non-commutative plane, see [22].

The exotic limit can be carried out also by letting $\alpha \rightarrow -\infty$, that involves the non-unitary representations \tilde{D}^j and \tilde{D}_α^+ . We reproduce here the same results as for $\alpha \rightarrow +\infty$. Putting $\alpha = -j$, $j \rightarrow \infty$, the limit can be applied, in particular, to usual boson/fermion fields.

Starting from Eqns. (3.16), (3.17), we now consider the exotic non-relativistic limit (9.1), i.e., $c \rightarrow \infty$, $j \rightarrow \infty$, $j/c^2 = \kappa = \text{const}$. The equations (9.39), with corresponding operators (9.40), are obtained by applying this limit to

$$(\kappa c)^{-1} V_0 \psi = 0, \quad V_+ \psi = 0, \quad -\kappa(2\kappa c^2)^{-1} V_- \psi = 0,$$

respectively. The oscillator number and creation-annihilation operators are found by applying the limit $j \rightarrow \infty$ to

$$\mathbf{N} = \mathcal{J}_0 + j + \epsilon, \quad b^\pm = \pm \frac{\mathcal{J}_\pm}{\sqrt{2(j + \epsilon)}}. \quad (9.52)$$

Note that while the nature of the operators (3.16) requires an indefinite metric, due to the sign in the definition of b^\pm in (9.52), we have now $\mathbf{N}^\dagger = \mathbf{N}$, $(b^+)^\dagger = b^-$ with respect to the usual, positive definite metric, and reproduce the relations (9.38).

10 Non-relativistic supersymmetry

Studying the non-relativistic limit of the vector equation (2.1), we derived first order equations for bosons and fermions, and for anyons. The symmetry of these systems is the mass-extended Galilei symmetry, or the exotic-Galilei symmetry, depending on the type of limit. They correspond to different Inönü-Wigner contractions of the Poincaré group.

In Sections 7 and 8, the relativistic wave equation (2.1) was generalized to (7.1), or (8.2), respectively. They describe $N = 1$ or $N = 2$ Poincaré supermultiplets. It is natural to expect, therefore, that the non-relativistic limits of these equations will describe (exotic) Galilei supermultiplets. It is also expected that the symmetries of these systems will be the Inönü-Wigner contraction of the super-Poincaré algebra.

In the case of $N = 1$ supersymmetry, for instance, the supermultiplet is described by (7.1). Projected to the corresponding subspaces, it yields Eqns. (7.6), $V_\mu^{(\alpha_+)} \psi^+ = 0$ and $V_\mu^{(\alpha_-)} \psi^- = 0$. The fields ψ^+ and ψ^- have spin α_+ and α_- , respectively. The procedure described in Section 9 to obtain the nonrelativistic limit can be repeated in each subspace. Having in mind that the fields ψ^\pm are expanded on the even (odd) Fock subspaces of the RDHA, see Eqns. (7.5) and (6.14), $\psi^\pm(x) = \sum_{n=0} \psi_n^\pm(x) |n\rangle_\pm$, we first obtain (9.3), (9.4). Then, taking the limit, yields Eqns. (9.21) and (9.22) with parameters α_+ and $\alpha_- = \alpha_+ + 1/2$, respectively. Each set of non-relativistic fields will be Galilei-symmetric, and the fields are interchanged by the non-relativistic supercharge, obtained as the non-relativistic limit of the relativistic supercharge (7.7).

The extended $N = 2$ supersymmetry is derived by a similar procedure, starting from equations (8.2) and applying Inönü-Wigner contraction. We present this in detail in the following subsections.

10.1 Usual $N = 1$ non-relativistic supermultiplet

In this case we get two copies of the non-relativistic equations (9.21), (9.22),

$$\sqrt{n + 2\alpha_\pm} \left(i \frac{\partial}{\partial t} \right) \phi_n^\pm + \sqrt{n + 1} P_+ \phi_{n+1}^\pm = 0, \quad (10.1)$$

$$\sqrt{n + 2\alpha_\pm} P_- \phi_n^\pm + 2m\sqrt{n + 1} \phi_{n+1}^\pm = 0. \quad (10.2)$$

Defining $\phi_n^\pm = c^n \psi_n^\pm$ and taking the limit $c \rightarrow \infty$ as in (9.21), (9.22), we get, cf. (9.25), (9.26),

$$\phi_n^\pm = (nB(2\alpha_\pm, n))^{-1/2} \left(\frac{-P_-}{2m} \right)^n \phi_0, \quad \phi_0^\pm = A^\pm \exp \left\{ -it \frac{\vec{p}^2}{2m} + i\vec{p}\vec{x} \right\}, \quad (10.3)$$

where the A^\pm are constants. The Galilean generators are defined as in (9.29), (9.30), (9.31). Augmented with the translation generators P_i , they span the algebra (9.32), (9.33). Here the reducible representations (6.5) of the Lorentz algebra (see also (6.17)) are used for \mathcal{J}_+ and \mathcal{J}_0 in the boost and rotation operators, \mathcal{K}_+ and \mathbf{J} , respectively. The operator \mathcal{C}_2 , defined as in (9.34), is therefore multi-valued. On-shell, we have

$$\mathcal{C}_2 = m\hat{\alpha}, \quad \mathcal{C}_2\Phi^\pm = \mathcal{C}_2^\pm\Phi^\pm, \quad \mathcal{C}_2^\pm = m\alpha_\pm. \quad (10.4)$$

The states Φ^\pm are vectors of the form (9.11), with components ϕ_n^\pm . The operator \mathcal{C}_1 vanishes in both subspaces.

The supercharges that interchange the fields Φ^\pm are the nonrelativistic limits of those in (7.7). To identify them, we note that the multi-component field Ψ of our $N = 1$ supermultiplet has the structure $\Psi^T = (\psi_0^+, \psi_0^-, \psi_1^+, \psi_1^-, \dots)$, where T denotes transposition. The rescaled field Φ has analogous structure. They are related by the transformation of the form (9.10) with a matrix \mathbf{M}

$$\mathbf{M} = \text{diag}(1, 1, c, c, c^2, c^2, \dots) = \mathbf{M}_+ + \mathbf{M}_-, \quad \mathbf{M}_\pm = \mathbf{M}\Pi_\pm, \quad (10.5)$$

where Π_+ and Π_- are the projectors (6.19). The similarity transformation (9.12), applied to the RDHA creation and annihilation operators, gives

$$\check{a}^+ = ca^+\Pi_- + a^+\Pi_+, \quad \check{a}^- = c^{-1}a^-\Pi_+ + a^-\Pi_-. \quad (10.6)$$

From (7.24) and (10.6) we infer the non-relativistic supercharges,

$$\mathcal{Q}_\pm = \lim_{c \rightarrow \infty} \frac{\check{Q}_\pm}{c}, \quad (10.7)$$

$$\mathcal{Q}_+ = 2i\sqrt{\frac{m}{1+\nu}} a^+\Pi_+, \quad \mathcal{Q}_- = -2i\sqrt{\frac{m}{1+\nu}} \left(a^- + \frac{1}{2m}P_-a^+ \right) \Pi_-, \quad (10.8)$$

where $\nu \neq -1$. The only nontrivial anticommutator is

$$\{\mathcal{Q}_+, \mathcal{Q}_-\} = \frac{8}{1+\nu} \mathcal{S}, \quad (10.9)$$

where

$$\mathcal{S} = \mathcal{C}_2 - m\hat{\alpha} + m\frac{1+\nu}{2} = \mathfrak{V}_0 + \frac{1}{2}m(1+\nu), \quad (10.10)$$

with \mathfrak{V}_0 defined in (9.17). The operator \mathcal{S} commutes with all Galilei generators. Unlike \mathcal{C}_2 , \mathcal{S} commutes also with the supercharges \mathcal{Q}_\pm . It can be identified [up to the factor m], therefore, with the superspin Casimir operator. Taking into account (10.4) [or the first equation from (9.20)], we get on-shell $\mathcal{S} = m\frac{1+\nu}{2}$. The supercharges \mathcal{Q}_\pm extend the Galilei algebra (9.32)-(9.33) with on-shell (anti)commutation relations,

$$[\mathbf{J}, \mathcal{Q}_\pm] = \pm \frac{1}{2}\mathcal{Q}_\pm, \quad \{\mathcal{Q}_+, \mathcal{Q}_-\} = 4m, \quad (10.11)$$

$$[\mathcal{K}_i, \mathcal{Q}_\pm] = [P_i, \mathcal{Q}_\pm] = [\mathcal{H}, \mathcal{Q}_\pm] = 0, \quad \mathcal{Q}_\pm^2 = 0, \quad (10.12)$$

giving rise to the $N = 1$ super-extension of the Galilei algebra [41]²¹.

²¹ The $N = 1$ super-Galilei algebra [41] has, besides the \mathcal{Q}_\pm , one more supercharge, namely the square root

10.2 Exotic $N = 1$ non-relativistic supermultiplet

Based on the deformed-oscillator representation of the Lorentz algebra, we define now the operators,

$$\mathbf{N} = \mathcal{J}_0 - \hat{\alpha}, \quad b^\pm = \sqrt{\frac{2}{\nu}} \mathcal{J}_\pm, \quad (10.13)$$

and take the limit $\nu \rightarrow \infty$. In this limit we reproduce the usual harmonic oscillator algebraic relations of the form (9.37),

$$[\mathbf{N}, b^\pm] = b^\pm, \quad [b^-, b^+] = 1. \quad (10.14)$$

However, here we get the direct sum of two irreducible representations of the Heisenberg algebra, realized in the subspaces \mathcal{F}_\pm of the total Fock space \mathcal{F} of the RDHA (see (6.14), (6.15)),

$$\mathbf{N}|n\rangle_\pm = n|n\rangle_\pm, \quad b^-|n\rangle_\pm = \sqrt{n+1}|n+1\rangle_\pm, \quad b^+|n\rangle_\pm = \sqrt{n}|n-1\rangle_\pm, \quad n = 0, 1, 2, \dots$$

Generalizing the exotic limit (9.1), we define, cf. (9.1),

$$c \rightarrow \infty, \quad \nu \rightarrow \infty, \quad \frac{\nu/4}{c^2} = \kappa = \text{const}. \quad (10.15)$$

This limit produces $\mathcal{J}_\pm/c \rightarrow \sqrt{2\kappa} b^\pm$ and,

$$\frac{\hat{\alpha}}{c^2} \rightarrow \kappa. \quad (10.16)$$

It follows that applying (10.15) to the supersymmetric equations (7.1) with the substitution (9.2) yields the *same* equations (9.39) and (9.42) as in the non-supersymmetric case. There, the v_\pm operators are defined as in (9.41), in terms of the operators (10.13).

The Poincaré reducibility of the system is revealed by a nontrivial integral, namely the reflection operator R . We have

$$\left(i\frac{\partial}{\partial t} - H\right)\psi^\pm = 0, \quad \Lambda_- \psi^\pm = 0, \quad R\psi^\pm = \pm\psi^\pm, \quad (10.17)$$

where the operators H and Λ_- , which commute with R , are formally given by the same relations as in (9.43), (9.40).

The finite part of the operator \mathcal{M}_0 is now

$$\mathbf{J} = (\epsilon_{ij}x_i P_j + \mathcal{J}_0) - \frac{\nu}{4} = \epsilon_{ij}x_i P_j + \mathbf{N} + \frac{1}{2} - \frac{1}{4}R. \quad (10.18)$$

The exotic Galilean generators are given by

$$\mathbf{J} = \epsilon_{ij}x_i P_j + \frac{1}{2}\kappa v_+ v_- + \frac{1}{2} - \frac{R}{4}, \quad \mathcal{K}_i = m x_i - t P_i + \kappa \epsilon_{ij} v_j. \quad (10.19)$$

Together with time and space translations, $\mathcal{H} = i\frac{\partial}{\partial t}$, P_i , they generate on-shell the algebra (9.49), (9.50). The Casimir operator \mathcal{C}_1 , defined in (9.51), vanishes on-shell. In turn, the operator \mathcal{C}_2 , (9.51), takes different eigenvalues when acts on the ψ^\pm wave functions,

$$\mathcal{C}_2 = m \frac{2-R}{4}, \quad \mathcal{C}_2 \psi^\pm = \mathcal{C}_2^\pm \psi^\pm, \quad \mathcal{C}_2^\pm = m \left(\frac{1}{2} \pm \frac{1}{4} \right). \quad (10.20)$$

of H . It is, however, *not* obtained as the non-relativistic limit of relativistic SUSY. The possibility of extending the Galilei symmetry conformally, yielding a generalized Schrödinger symmetry from $s = 1/2$ [42] to any spin, as well as its supersymmetric extensions, [43], are, in general, open questions. (See, however, [44]). The extension of exotic symmetry appears to be non-linear [28, 45].

The supercharges associated with the system (10.17) are obtained from the exotic limit of (7.24),

$$\mathcal{Q}_\pm = \lim_{c, \nu \rightarrow \infty} \frac{1}{\sqrt{c}} Q_\pm = \pm 2i\sqrt{m} \left(\lim_{\nu \rightarrow \infty} \Pi_\mp \frac{a^\pm}{\sqrt{\nu}} \right). \quad (10.21)$$

On account of the realization (6.29) of the RDHA generators in terms of the bosonic oscillator operators, we get

$$\lim_{\nu \rightarrow \infty} \frac{a^+}{\sqrt{\nu}} = \frac{1}{\sqrt{\tilde{\mathcal{N}}}} \tilde{a}^+ \tilde{\Pi}_+ \equiv c^+, \quad \lim_{\nu \rightarrow \infty} \frac{a^-}{\sqrt{\nu}} = \frac{1}{\sqrt{1 + \tilde{\mathcal{N}}}} \tilde{a}^- \tilde{\Pi}_- \equiv c^-. \quad (10.22)$$

The operators c^\pm and $\tilde{R} = R$ satisfy the same algebra as the Pauli matrices $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ and σ_3 ,

$$\{c^+, c^-\} = 1, \quad c^{\pm 2} = 0, \quad [c^+, c^-] = R, \quad \{c^\pm, R\} = 0, \quad R^2 = 1. \quad (10.23)$$

We obtain therefore

$$\mathcal{Q}_\pm = \pm 2i\sqrt{m} c^\pm. \quad (10.24)$$

The supercharges \mathcal{Q}_\pm extend the exotic Galilei algebra (9.49)-(9.50) with (anti)commutation relations identical to those in (10.11)-(10.12). The only difference with the usual non-relativistic limit of the previous subsection is the non-commutativity of Galilean boosts.

The Casimir operator (10.20) is generalized to

$$\mathcal{S} = \mathcal{C}_2 + \frac{1}{16}[\mathcal{Q}_+, \mathcal{Q}_-],$$

that commutes, unlike \mathcal{C}_2 , with all superalgebra generators, including the supercharges \mathcal{Q}_\pm , and on-shell takes the value $\frac{1}{2}m$. It is identified [up to the factor m] as the superspin operator.

10.3 Nonrelativistic Dirac/DJT supermultiplet

- For spin one-half ($-\alpha = 1/2$), putting

$$\phi_0 = \psi_0, \quad \phi_1 = c\psi_1, \quad (10.25)$$

the general theory of Section 9 reduces, for $n = 0$, to the non-relativistic analogs of the (2+1)D Dirac equation due to the Lévy-Leblond [21, 42], which take here a form

$$\begin{aligned} i \frac{\partial}{\partial t} \phi_0 + -iP_+ \phi_1 &= 0, \\ P_- \phi_0 - 2im\phi_1 &= 0. \end{aligned} \quad (10.26)$$

Eliminating one component yields the Schrödinger equation for the other one. Eqns. (9.21)–(9.22) for $n = 1$ and then for any $n > 1$ yield $\phi_n = 0$.

The boosts operators (9.29) involve nontrivial spin [21, 42],

$$\mathcal{K}_\pm = \mathcal{K}_1 \pm i\mathcal{K}_2 = -tP_\pm + mx_\pm + \Delta_\pm, \quad \Delta_- = 0, \quad \Delta_+ = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \quad (10.27)$$

The angular momentum, (9.30), reads in turn,

$$\mathbf{J} = \epsilon_{ij} x_i P_j + \mathcal{J}_0, \quad \mathcal{J}_0 = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}. \quad (10.28)$$

• For $-\alpha = j = 1$ one gets instead the non-relativistic analog of the DJT vector equation for a topological massive gauge field. Introducing the redefined fields

$$\varphi_0 = \psi_0, \quad \varphi_1 = c\psi_1, \quad \varphi_2 = c^2\psi_2, \quad (10.29)$$

yields, putting $n = 0$ into (9.21)-(9.22) and multiplying by $-i$, the pair of equations

$$\begin{aligned} \sqrt{2}i\frac{\partial}{\partial t}\varphi_0 - iP_+\varphi_1 &= 0, \\ \sqrt{2}P_-\varphi_0 - 2im\varphi_1 &= 0. \end{aligned}$$

Similarly, for $n = 1$ we get

$$\begin{aligned} P_-\varphi_1 - i2\sqrt{2}m\varphi_2 &= 0, \\ i\frac{\partial}{\partial t}\varphi_1 - i\sqrt{2}P_+\varphi_2 &= 0. \end{aligned}$$

The last equation here is readily seen to follow from the first three, however, and can therefore be dropped. For $n = 2$ and then for any $n > 2$, the system (9.21)-(9.22) yields $\phi_n = 0$. In conclusion, the non-relativistic limit of the Deser-Jackiw-Templeton equations (2.22) reads,

$$\begin{aligned} i\frac{\partial}{\partial t}\varphi_0 - \frac{i}{\sqrt{2}}P_+\varphi_1 &= 0, \\ i\sqrt{2}m\varphi_1 - P_-\varphi_0 &= 0, \\ 2\sqrt{2}m\varphi_2 + iP_-\varphi_1 &= 0. \end{aligned} \quad (10.30)$$

φ_1 and φ_2 play the role of auxiliary fields, and may be expressed in terms of φ_0 . The dynamics of each of these free fields is governed by the Schrödinger equation.

The spin contribution to the boost, (9.29), and to the angular momentum, (9.30), read

$$\Delta_+ = \begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \end{pmatrix}, \quad \mathcal{J}_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (10.31)$$

The Lévy-Leblond (spin $s = -1/2$) and non-relativistic DJT (spin $s = -1$) systems can now be unified into an $N = 1$ supermultiplet following the general recipe of Section 10. The non-relativistic spin one half, (10.25), and spin one, (10.29), multiplets are thus unified into

$$\Phi = \begin{pmatrix} \varphi_0 \\ \phi_0 \\ \varphi_1 \\ \phi_1 \\ \varphi_2 \end{pmatrix}. \quad (10.32)$$

The Galilei generators P_i and (9.30) with $\mathcal{J}_0 = \text{diag}(-1, -1/2, 0, 1/2, 1)$ [cf. (A.11)] act diagonally on the vectors (10.32). The corresponding supercharges are (10.8) with $\nu = -5$, i.e.

$$\mathcal{Q}_+ = \sqrt{m}a^+\Pi_+, \quad \mathcal{Q}_- = -\sqrt{m}\left(a^- + \frac{1}{2m}P_-a^+\right)\Pi_-, \quad (10.33)$$

where the odd $\mathfrak{osp}(1|2)$ generators a^\pm and projectors Π_\pm are those matrices (A.10) and (A.9). Note the asymmetry between \mathcal{Q}_+ and \mathcal{Q}_- , which is consistent with the one between \mathcal{K}_+ and \mathcal{K}_- in Eqns. (10.27), (10.31).

The Dirac – DJT supermultiplet, in particular its superGalilei symmetry, are further discussed in [44]. See also [37].

11 Extended $N = 2$ non-relativistic supersymmetry

Here we investigate the non-relativistic limit of the relativistic equations (8.16) with $N = 2$ supersymmetry. The symmetries of the corresponding four-particle non-relativistic supermultiplet are obtained by Inönü-Wigner contraction of the extended superPoincaré algebra (7.10), (8.20)-(8.21). The limit can be usual, or exotic, or a mixture of both kinds, since there are two independent spin parameters.

11.1 Usual non-relativistic extended supersymmetry

First we consider the usual non-relativistic limit of the four-anyon system in Eqns.(8.16),

$$\begin{aligned} V_\mu^{(\chi)} \psi^{++}(x) &= 0, \\ V_\mu^{(\chi+1/2)} \psi^{+-}(x) &= 0, \\ V_\mu^{(\chi+1/2)} \psi^{-+}(x) &= 0, \\ V_\mu^{(\chi+1)} \psi^{--}(x) &= 0. \end{aligned}$$

For $\psi^{++}(x)$, for instance, the simple limit yields

$$\sqrt{n+2\chi} \left(i \frac{\partial}{\partial t} \right) \Phi_n^{++} + \sqrt{n+1} P_+ \Phi_{n+1}^{++} = 0, \quad (11.1)$$

$$\sqrt{n+2\chi} P_- \Phi_n^{++} + 2m\sqrt{n+1} \Phi_{n+1}^{++} = 0, \quad (11.2)$$

obtained as in (10.1), (10.2). Identical equations can be derived for the three remaining fields, taking into account that the spin parameter in $V_\mu^{(\hat{\beta})}$ takes the values,

$$\hat{\beta} = \text{diag} \left(\chi, \chi + \frac{1}{2}, \chi + \frac{1}{2}, \chi + 1 \right), \quad \chi = \alpha_{1+} + \alpha_{2+} = \frac{1}{4}(1 + \nu_1) + \frac{1}{4}(1 + \nu_2), \quad (11.3)$$

as in (8.11). The index n of Φ refers to the lowest weight representations of the Lorentz algebra of spin β , from which the nonrelativistic limit was taken. These lowest weight representations correspond to linear combinations of the Fock basis (8.14) (see also (8.12) and (8.13)). Hence,

$$\hat{\beta} = \hat{\alpha}_{1+} + \hat{\alpha}_{2+} = \chi + \frac{1}{2} (\Pi_{1-} + \Pi_{2-}), \quad \chi = \alpha_{1+} + \alpha_{2+} = \frac{1}{2} + \frac{1}{4}(\nu_1 + \nu_2). \quad (11.4)$$

The nonrelativistic limit of time translations, Galilei boosts and rotation generators is obtained as in Section 9.

This representation is, of course, reducible, since that of the Poincaré algebra was reducible too. In particular, rotations also involve the number operators of the lowest-weight representations of the Lorentz algebra as well as $\hat{\beta}$,²²

$$\mathbf{J} = \epsilon_{ij} x_i P_j + \mathbf{N}_1 + \mathbf{N}_2 + \hat{\beta}. \quad (11.5)$$

²² Here $\Pi_{\underline{A}\pm} = (1 \pm R_{\underline{A}})/2$ are the projection operators on Fock space of every oscillator labeled by \underline{A} . They satisfy the relations (6.19) for every index, and commute. The numbers operators satisfy,

$$\mathbf{N}_{\underline{A}} |n_1, n_2\rangle_{\pm\pm} = n_A |n_1, n_2\rangle_{\pm\pm}, \quad \mathbf{N}_{\underline{A}} |n_1, n_2\rangle_{\pm\mp} = n_A |n_1, n_2\rangle_{\pm\mp}, \quad A = 1, 2.$$

The operator \mathcal{C}_1 in (9.35) is still zero, while \mathcal{C}_2 becomes $\mathcal{C}_2 = m\hat{\beta}$, cf. Eq. (9.35). Their eigenvalues are,

$$\begin{aligned}\mathcal{C}_2\Phi^{++} &= \chi\Phi^{++}, & \mathcal{C}_2\Phi^{+-} &= (\chi + 1/2)\Phi^{+-}, \\ \mathcal{C}_2\Phi^{-+} &= (\chi + 1/2)\Phi^{-+}, & \mathcal{C}_2\Phi^{--} &= (\chi + 1)\Phi^{--}.\end{aligned}\tag{11.6}$$

Each field Φ carries therefore a representation of the Galilei algebra.

The supercharges associated to this supermultiplet are the nonrelativistic limits of the supercharges (8.17) and (8.18), namely,

$$\mathcal{Q}_{1+} = 2i\sqrt{\frac{m}{1+\nu_1}}a_1^+\Pi_{1+}, \quad \mathcal{Q}_{1-} = -2i\sqrt{\frac{m}{1+\nu_1}}\left(a_1^- + \frac{1}{2m}P_-a_1^+\right)\Pi_{1-}, \tag{11.7}$$

$$\mathcal{Q}_{2+} = 2i\sqrt{\frac{m}{1+\nu_2}}R_1a_2^+\Pi_{2+}, \quad \mathcal{Q}_{2-} = -2i\sqrt{\frac{m}{1+\nu_2}}R_1\left(a_2^- + \frac{1}{2m}P_-a_2^+\right)\Pi_{2-}, \tag{11.8}$$

cf. (10.8). The algebra generated by the supercharges (11.7), (11.8) augmented with the Galilei generators is analogous to (10.11), (10.12), with the additional relations

$$\{\mathcal{Q}_{A+}, \mathcal{Q}_{B-}\} = \delta_{AB}4m, \quad \{\mathcal{Q}_{A\pm}, \mathcal{Q}_{B\pm}\} = 0. \tag{11.9}$$

The supercharges with different indices anticommute, due to the presence of the reflection operator, R_1 .

11.2 Exotic nonrelativistic extended supersymmetry

The exotic limit (9.1) is generalized to

$$c \rightarrow \infty, \quad \nu \rightarrow \infty, \quad \frac{\hat{\beta}}{c^2} = \frac{\chi}{c^2} = \kappa = \text{const}, \tag{11.10}$$

cf. (10.16). Since χ depends on both of the deformation parameters (see (11.5)), the exotic limit can be carried out varying only one, or both, parameters simultaneously.

Simple exotic limit

Here we take the limit

$$c \rightarrow \infty, \quad \nu_1 \rightarrow \infty, \quad \nu_2 = \text{const}, \quad \frac{\nu_1/4}{c^2} = \kappa = \text{const}. \tag{11.11}$$

The rotation generator is obtained by removing the divergent c -number term $\nu_1/4$ from $\hat{\beta}$ in (11.5),

$$\mathbf{J} = \epsilon_{ij}x_iP_j + \mathbf{N}_1 + \mathbf{N}_2 + \alpha_{2+} + \frac{1}{4} + \frac{1}{2}(\Pi_{1-} + \Pi_{2-}). \tag{11.12}$$

Galilei boosts are obtained from the non-relativistic limit of Sec. 9,

$$\mathcal{K}_+ = mx_+ - tP_+ + i(\mathcal{J}_{2+} - \kappa v_+), \quad \mathcal{K}_- = mx_- - tP_- + i\kappa v_-, \tag{11.13}$$

$$v_{\pm} = v_1 \pm iv_2 = -\sqrt{\frac{2}{\kappa}}b^{\pm}, \quad b^{\pm} = \lim_{\nu_1 \rightarrow \infty} \sqrt{\frac{2}{\nu_1}}\mathcal{J}_{1\pm}. \tag{11.14}$$

Observe that v_j and b^\pm only involve the oscillator with label $\underline{1}$.

Galilei boosts satisfy the exotic commutation relation (9.49). The b^\pm and \mathbf{N}_\pm operators satisfy relations like in (9.37), *i.e.* $\mathbf{N}_\pm = b^\pm b^\mp$. Due to the asymmetry of the limit (11.11), the supercharges are of different types,

$$\mathcal{Q}_{\pm\pm} = \lim_{c, \nu \rightarrow \infty} \frac{1}{\sqrt{c}} Q_{\pm\pm}, \quad \mathcal{Q}_{2\pm} = \lim_{c \rightarrow \infty} \frac{1}{c} \check{Q}_{2\pm}, \quad (11.15)$$

$$\mathcal{Q}_{1\pm} = \pm 2i\sqrt{m} \left(\lim_{\nu_1 \rightarrow \infty} \Pi_{1\mp} \frac{a_1^\pm}{\sqrt{\nu_1}} \right) = \pm 2i\sqrt{m} c_1^\pm, \quad (11.16)$$

$$\mathcal{Q}_{2+} = 2i \sqrt{\frac{m}{1+\nu_2}} R_1 a_2^+ \Pi_{2+}, \quad \mathcal{Q}_{2-} = -2i \sqrt{\frac{m}{1+\nu_2}} R_1 \left(a_2^- + \frac{1}{2m} P_- a_2^+ \right) \Pi_{2-}, \quad (11.17)$$

which are those obtained as in Eqns. (10.21) (“exotic supercharges”) and (10.8) (“usual supercharges”) respectively. These supercharges yield commutation relation as in (11.9).

Double exotic limit

Here we take the limit,

$$c \rightarrow \infty, \quad \nu_1 \rightarrow \infty, \quad \nu_2 \rightarrow \infty, \quad \frac{\nu_1/2}{c^2} = \frac{\nu_2/2}{c^2} = \kappa = \text{const}. \quad (11.18)$$

The finite part of angular momentum is now,

$$\mathbf{J} = \epsilon_{ij} x_i P_j + \mathbf{N}_1 + \mathbf{N}_2 + \frac{1}{2} + \frac{1}{2} (\Pi_{1-} + \Pi_{2-}). \quad (11.19)$$

The Galilei boosts are,

$$\mathcal{K}_i = m x_i - t P_i + \frac{\kappa}{2} \epsilon_{ij} v_{1j} + \frac{\kappa}{2} \epsilon_{ij} v_{2j} \quad (11.20)$$

where the v operators are defined analogously to (11.2), and the labels \underline{A} indicate the dependence on the corresponding oscillator. In particular, here we have $b_{\underline{A}}^\pm$ and $\mathbf{N}_{\underline{A}} = b_{\underline{A}}^+ b_{\underline{A}}^-$, as in (9.37). Note that this double exotic limit is symmetric in both oscillators. Hence, we obtain,

$$\mathcal{Q}_{1\pm} = \lim_{c, \nu_1 \rightarrow \infty} \frac{1}{\sqrt{c}} Q_{1\pm} = \pm 2i\sqrt{m} c_1^\pm, \quad (11.21)$$

$$\mathcal{Q}_{2\pm} = \lim_{c, \nu_2 \rightarrow \infty} \frac{1}{\sqrt{c}} Q_{2\pm} = \pm 2i\sqrt{m} R_1 c_2^\pm, \quad (11.22)$$

which are those in Eqns. (10.21) and (10.8) respectively.

12 Discussion and outlook

In conclusion, we review our main results and hint at some open problems which could be worth of further study.

Our universal covariant vector equations describe massive spinning particles in $2+1$ dimensions. The Poincaré spin, a pseudoscalar, can take arbitrary real values. The equations fix themselves the type of the representation of the $(2+1)\text{D}$ Lorentz algebra: only $\mathfrak{so}(2, 1)$ representations bounded from below or from above (or from both sides) are allowed. Depending on the spin parameter, α , our equations describe three classes of particles, namely

- a) Bosons/Fermions : $\alpha = -j$, $2j = 1, 2, \dots$,
- b) “Unitary” anyons : $\alpha > 0$,
- c) “Non-unitary” anyons : $-j < \alpha < -j + 1/2$.

Case a), based on finite-dimensional non-unitary representations of the $\mathfrak{so}(2, 1)$, reproduces, for $\alpha = -1/2, -1, -3/2$ the Dirac, Deser-Jackiw-Templeton, and Rarita-Schwinger theories, respectively. For $\alpha = -2$, our theory provides us with the linearized equations of massive gravity. More generally, complete correspondence with the $2+1$ dimensional Dirac-Fierz-Pauli equations [46] is expected.

Then, we investigated two types of anyons. Both involve infinite-dimensional half-bounded representations of the $(2+1)\text{D}$ Lorentz algebra. The first type, which corresponds to *unitary* representations [case b) above], has been widely used in the anyon context since the beginning [13, 14, 15]. The second type [case c)] is associated with *non-unitary* representations. No attention was paid to them in earlier investigations on anyons. We argue, however, that precisely these new- [and not the conventional] types of fractional spin particles do correspond to the intuitive picture, in which anyons interpolate between bosons and fermions.

The essential difference is that when the parameter α tends to a (half) integer $-j$, the higher field components $\psi_n(x)$ with index $n > 2j$ are, for the new type, suppressed by the universal numerical factor $(\alpha + j)^{1/2}$. Note that no such a suppression happens in the conventional, unitary case. As a result, in the limiting case $\alpha = -j$, such an anyonic field reduces to a usual $(2j + 1)$ -component field of spin modulus $|s| = j$. The spin zero case then also can be incorporated into the theory letting $\alpha \rightarrow 0$ for non-unitary anyons.

The physical difference between the two types of anyons could, perhaps, be revealed when interactions are switched on.

The extended formulation of the theory, presented in Section 5, generalizes the Jackiw-Nair [13] and the Majorana-Dirac [14] descriptions of anyons. Our framework allows us to combine several irreducible representations of the Lorentz algebra, “entangled” by Eqn. (5.3), whereas spins behave additively. Usual bosons and fermions can be obtained, in particular, as *entangled* anyons. The topologically massive vector gauge field of Jackiw-Deser-Templeton [3] can be, for example, viewed as a pair of entangled Dirac particles.

It would be interesting to apply such a picture to quantum computing [47].

Promoting the parameter α to a diagonal operator $\hat{\alpha} = \text{diag}(\alpha, \alpha + 1/2)$ and taking the direct sum of two irreducible representations of $\mathfrak{so}(2, 1)$ provided us with an $N = 1$ supersymmetric system. It is described, *on-shell*, by the usual super-Poincaré algebra.

The natural question is, then: what kind of symmetry will appear if the elements of $\hat{\alpha}$ are shifted by $n/2$ with n integer greater than 1? One can expect that such a theory will be characterized by a kind of generalized supersymmetry, with spin-tensorial supercharges [48, 49, 50], which, on-shell, would generate a *nonlinear* generalization of the usual $N = 1$ super-Poincaré algebra [35, 51].

In the simplest form of the extended realization two Lorentz spin vector generators are added; this also allows us to generalize the $N = 1$ supersymmetry to $N = 2$, see Section 8 for details.

The extended formulation can further be generalized by adding an arbitrary number of the vector spin generators, $\mathcal{J}_\mu = \sum_i J_\mu^{(i)}$, with each $J_\mu^{(i)}$ belonging to one of the representations $\mathcal{D}_{\alpha_{(i)}}^+$ (or $\mathcal{D}_{\alpha_{(i)}}^-$) from (2.19). Choosing $\beta = \sum_i \alpha_{(i)}$, one finds then that the basic vector set of equations splits into a set of vector equations, while the irreducibility equation (2.10) produces

an entangling equation which generalizes (5.3),

$$V_\mu^{(\alpha_{(i)})}\psi = 0, \quad \sum_{i>j} (J^{(i)}J^{(j)} + \alpha_{(i)}\alpha_{(j)})\psi = 0. \quad (12.1)$$

Each $J_\mu^{(i)}$ can be promoted to an even generator of the $\mathfrak{osp}(1|2)$ superalgebra, realized in terms of an independent reflection-deformed oscillator with appropriately chosen parameter $\nu_{(i)}$. In this way, extended supermultiplets of anyons, or of bosons and fermions could be obtained.

Considering two types of non-relativistic limits to the relativistic theory, we observed how either the usual centrally-mass-extended, or the doubly centrally extended “exotic” Galilei symmetries appear. Their Schrödinger-type conformal extensions for arbitrary (including anyonic) spin, and also for supersymmetric generalizations could also be studied.

It would also be interesting to derive our basic vector equations from an action principle. A nontrivial point is that the three equations of our system are dependent: any two of them generate the third one as consistency (integrability) equation. This indicates that the corresponding action should have a kind of gauge symmetry, analogous to the Chern-Simons formulation of topologically massive gauge fields [3]. An additional indication in this direction is the appearance of an infinite-dimensional null subspace in the extended formulation. Analogous null subspaces also appear, in fact, in covariant quantization schemes of relativistic strings, and are associated with gauge (diffeomorphism) symmetries.

In our unified scheme, all irreducible massive representations of the 2+1 D Poincaré group have been obtained. Does this theory have an analog in higher dimensions? This question directs us towards theories with arbitrary spin fields. For instance, to strings, or to higher spin theories [52].

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A Finite-dimensional representations of $\mathfrak{osp}(1|2)$

A.1 $\mathfrak{osp}(1|2)$ -supermultiplet ($\alpha_+ = -1/2$, $\alpha_- = 0$)

Here $\nu = -3$ and $r = 1$. The finite-Fock space, $\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-$, is

$$\mathcal{F}_+ = \left\{ |0\rangle_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |1\rangle_+ = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \mathcal{F}_- = \left\{ |0\rangle_- = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}. \quad (A.1)$$

The reflection operator and projectors take the form,

$$R = \text{diag}(1, -1, 1), \quad \Pi_+ = \text{diag}(1, 0, 1), \quad \Pi_- = \text{diag}(0, 1, 0). \quad (A.2)$$

The matrix form of the $\mathfrak{osp}(1|2)$ algebra is obtained from (6.22)–(6.25), (6.15) and (6.16),

$$a^+ = \begin{pmatrix} 0 & 0 & 0 \\ i\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \quad a^- = \begin{pmatrix} 0 & i\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad (A.3)$$

$$\mathcal{J}_0 = \begin{pmatrix} -1/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \quad \mathcal{J}_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_- = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.4})$$

The representation (A.4) of the Lorentz algebra is reducible. The spin $-\alpha_+ = j = 1/2$ part is projected from (A.4) to

$$J_0^{(+)} = -\frac{1}{2}\sigma_3, \quad J_1^{(+)} = \frac{i}{2}\sigma_1, \quad J_2^{(+)} = -\frac{i}{2}\sigma_2, \quad J_\mu^{(+)} J^{(+)\mu} = -\frac{3}{4}, \quad (\text{A.5})$$

where σ_a , $a = 1, 2, 3$, are the Pauli matrices. These operators act on the vector space \mathcal{F}_+ formed by

$$|0\rangle_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_+, \quad |1\rangle_+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_+.$$

$-2J_\mu^{(+)} = \gamma'_\mu$ are Dirac matrices that satisfy the Clifford algebra (6.10). They are related to the Majorana representation (6.9) by the unitary transformation,

$$\gamma'_\mu = U \gamma_\mu U^\dagger, \quad U = \frac{1}{\sqrt{2}}(\sigma_2 + \sigma_3). \quad (\text{A.6})$$

The Lorentz algebra (A.4) projected to the spin $\alpha_- = 0$ subspace is trivial, $J_\mu^{(-)} = \mathcal{J}_\mu \Pi_- = 0$. It leaves invariant the scalar, one-dimensional subspace $\mathcal{F}_- = \{|0\rangle_-\}$.

A.2 $\mathfrak{osp}(1|2)$ -supermultiplet ($\alpha_+ = -1$, $\alpha_- = -1/2$)

Here $\nu = -5$ and $r = 2$. The finite-Fock space is $\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-$,

$$\mathcal{F}_+ = \left\{ |0\rangle_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |1\rangle_+ = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |2\rangle_+ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad (\text{A.7})$$

$$\mathcal{F}_- = \left\{ |0\rangle_- = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |1\rangle_- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}. \quad (\text{A.8})$$

The reflection operator and projectors take the form

$$R = \text{diag}(1, -1, 1, -1, 1), \quad \Pi_+ = \text{diag}(1, 0, 1, 0, 1), \quad \Pi_- = \text{diag}(0, 1, 0, 1, 0). \quad (\text{A.9})$$

The $\mathfrak{osp}(1|2)$ matrix algebra is generated by

$$a^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}, \quad a^- = \begin{pmatrix} 0 & 2i & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.10})$$

$$\mathcal{J}_0 = \text{diag}(-1, -1/2, 0, 1/2, 1), \quad (\text{A.11})$$

$$\mathcal{J}_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{2} & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_- = \begin{pmatrix} 0 & 0 & i\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & i\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.12})$$

Projection to the odd subspace of spin $-\alpha_- = j = 1/2$ produces a representation with the same matrix elements as in (A.5) but now acting on \mathcal{F}_- ,

$$|0\rangle_- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_-, \quad |1\rangle_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_-. \quad (\text{A.13})$$

The spin $-\alpha_+ = j = 1$ part is projected to

$$J_0^{(+)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad J_1^{(+)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad J_2^{(+)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (\text{A.14})$$

which act on the vector space \mathcal{F}_+ ,

$$|0\rangle_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_+, \quad |1\rangle_+ = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_+, \quad |2\rangle_+ = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_+. \quad (\text{A.15})$$

(A.14) is related to the adjoint representation of $\mathfrak{so}(2, 1)$,

$$(J_\mu)^\nu{}_\lambda = -i\epsilon^\nu{}_{\mu\lambda}, \quad \mu, \nu, \lambda = 0, 1, 2, \quad \epsilon^{012} = 1, \quad (\text{A.16})$$

by means of the unitary transformation

$$J_\mu^{(+)} = U J_\mu U^\dagger, \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & i \\ -i\sqrt{2} & 0 & 0 \\ 0 & -1 & i \end{pmatrix}. \quad (\text{A.17})$$

The explicit matrix form of (A.16) is

$$(J_0)^\nu{}_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad (J_1)^\nu{}_\lambda = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad (J_2)^\nu{}_\lambda = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.18})$$

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